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Adaptive Learning Structures for Real-Time Optimal Control and Dynamic Games

Talk available online at
http://ARRI.uta.edu/acs

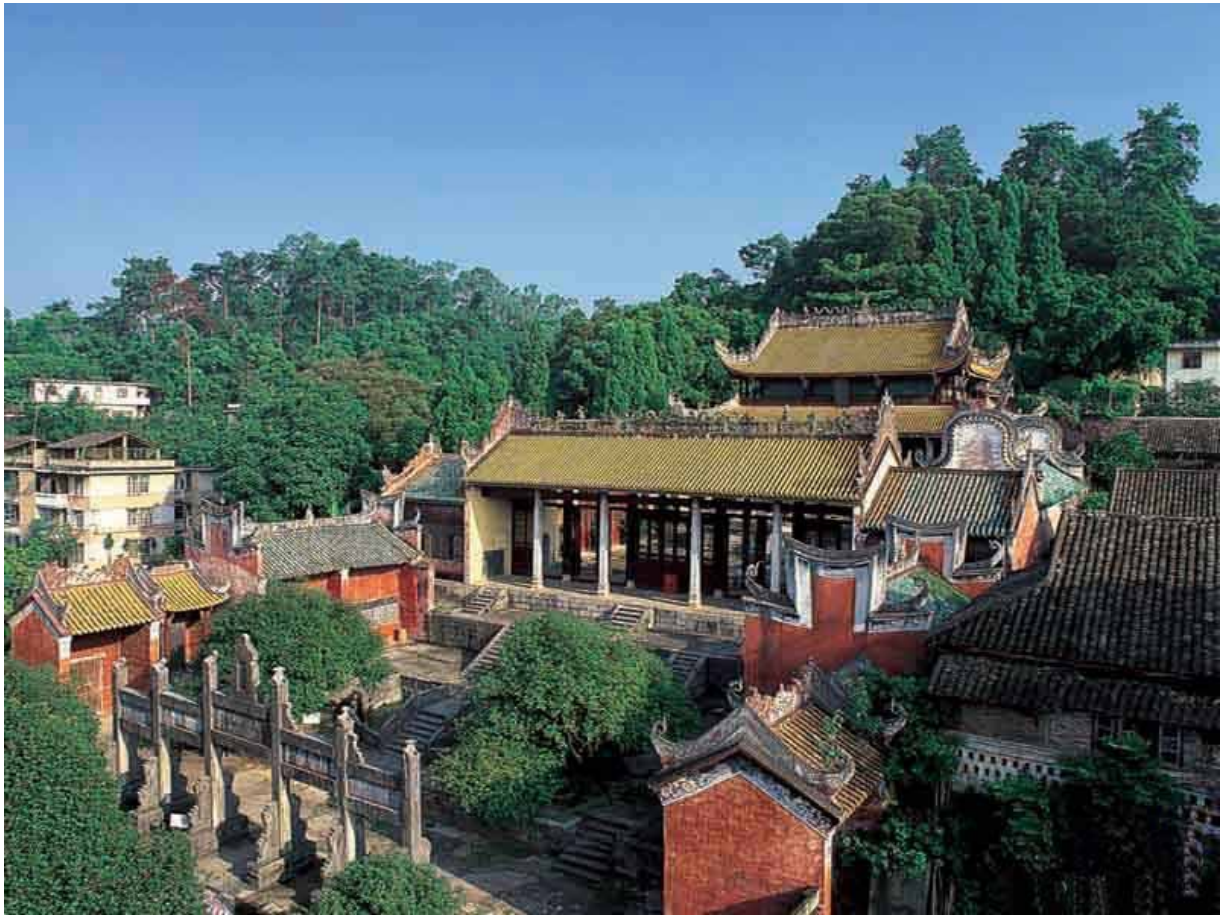
Meng Tze
500 BC

Mencius

He who exerts his mind to the utmost knows nature's pattern.

The way of learning is none other than finding the lost mind.

Man's task is to understand patterns in nature and society.



- Optimal Control
- Reinforcement learning
- Integral Reinforcement Learning for Continuous-time systems
- Policy Iteration for ZS games
- Synchronous Policy Iteration
- PI for Solving Dynamic Games online

It is man's obligation to explore the most difficult questions in the clearest possible way and use reason and intellect to arrive at the best answer.

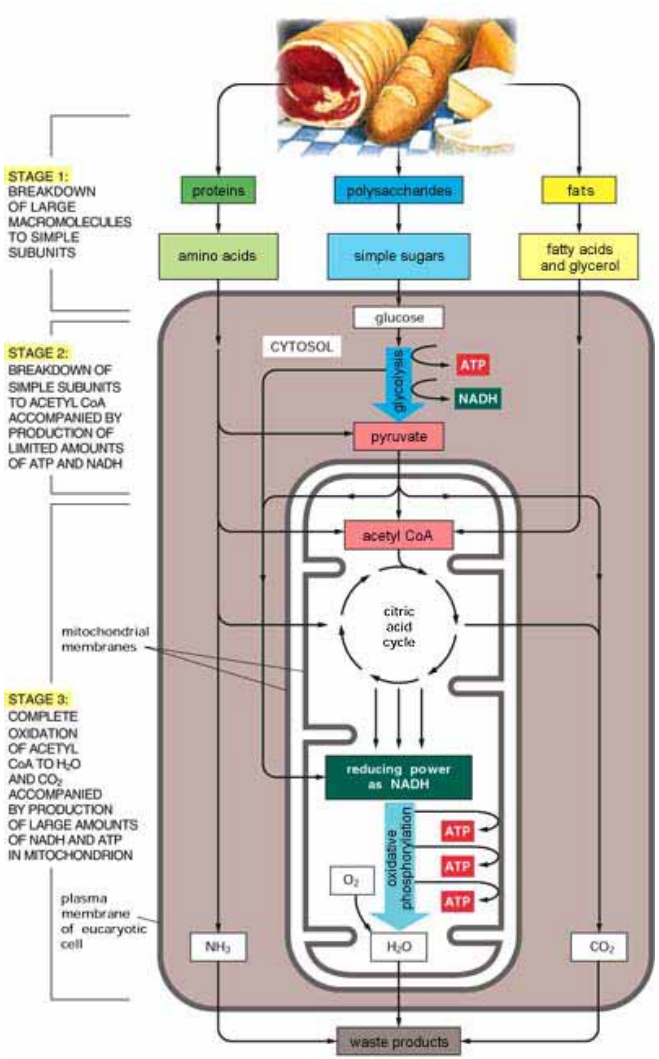
Man's task is to understand patterns in nature and society.

The first task is to understand the individual problem, then to analyze symptoms and causes, and only then to design treatment and controls.

Ibn Sina 1002-1042
(Avicenna)

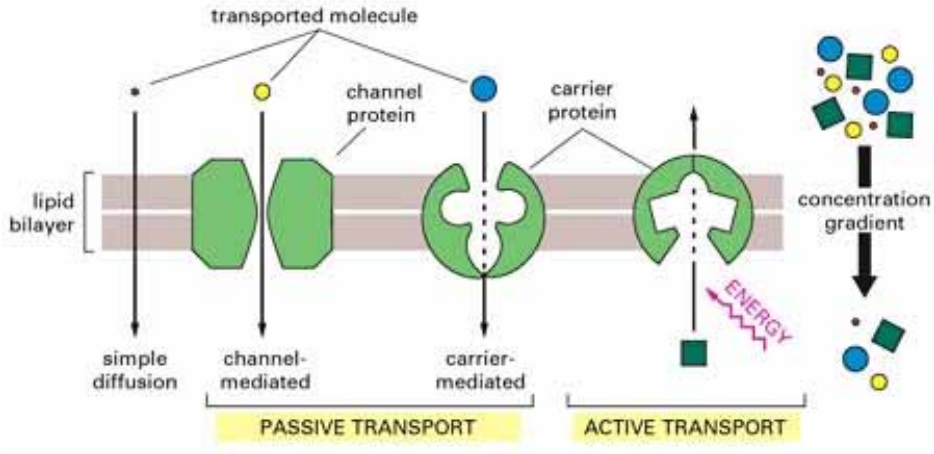
Optimality in Biological Systems

Cell Homeostasis



Cellular Metabolism

The individual cell is a complex feedback control system. It pumps ions across the cell membrane to maintain homeostatis, and has only limited energy to do so.



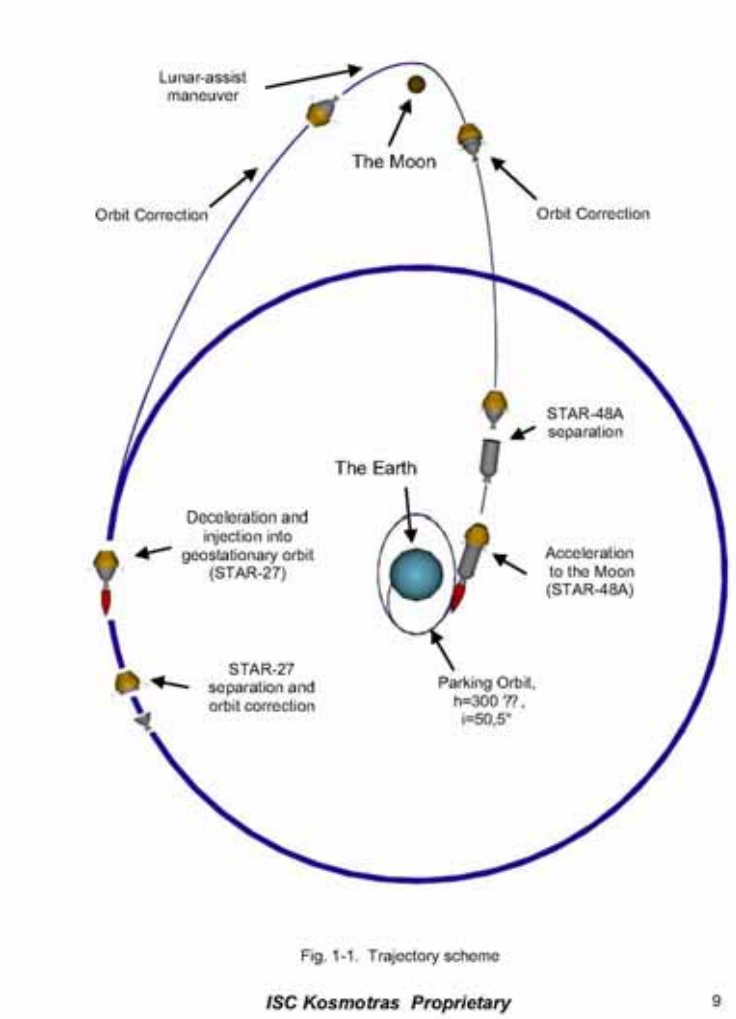
Permeability control of the cell membrane

http://www.accessexcellence.org/RC/VL/GG/index.html

R. Kalman 1960

Optimality in Control Systems Design

Rocket Orbit Injection



http://microsat.sm.bmstu.ru/e-library/Launch/Dnepr_GEO.pdf

Dynamics

$$\dot{r} = w$$

$$\dot{w} = \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{F}{m}\sin\phi$$

$$\dot{v} = \frac{-wv}{r} + \frac{F}{m}\cos\phi$$

$$\dot{m} = -Fm$$

Objectives

- Get to orbit in minimum time
- Use minimum fuel

Optimality and Games

Optimal Control is Effective for:

- Aircraft Autopilots
- Vehicle engine control
- Aerospace Vehicles
- Ship Control
- Industrial Process Control

Multi-player Games Occur in:

- Economics
- Control Theory disturbance rejection
- Team games
- International politics
- Sports strategy

But, optimal control and game solutions are found by

- Offline solution of Matrix Design equations
- A full dynamical model of the system is needed

We want to find optimal control solutions online in real-time using adaptive control techniques

Adaptive Control Structures for:

A. Optimal control B. Zero-sum games C. Non zero-sum games

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1. System dynamics
2. Value/cost function
3. Bellman equation
4. HJ solution equation (Riccati eq.)
5. Policy iteration – gives the structure we need

Books

F.L. Lewis, D. Vrabie, and V. Syrmos,
Optimal Control, third edition, John Wiley and Sons, New York, 2012.

New Chapters on:
Reinforcement Learning
Differential Games

D. Vrabie, K. Vamvoudakis, and F.L. Lewis, *Optimal Adaptive Control and Differential Games by Reinforcement Learning Principles*, IET Press, 2012, to appear.

Continuous-Time Optimal Control

System dynamics $\dot{x} = f(x,u) = f(x) + g(x)u$

Cost/value $V(x(t)) = \int_t^\infty r(x,u)\,dt = \int_t^\infty (Q(x) + u^T R u)\,dt$

Leibniz gives Differential equivalent

Bellman Equation, in terms of the Hamiltonian function

$$H(x, \frac{\partial V}{\partial x}, u) = \dot{V} + r(x,u) = \left(\frac{\partial V}{\partial x}\right)^T \dot{x} + r(x,u) = \left(\frac{\partial V}{\partial x}\right)^T \left(f(x) + g(x)u\right) + r(x,u) = 0$$

Stationarity condition $\frac{\partial H}{\partial u} = 0$

Stationary Control Policy $u = h(x) = -\frac{1}{2} R^{-1} g^T(x) \frac{\partial V}{\partial x}$

HJB equation $0 = \left(\frac{dV^*}{dx}\right)^T f + Q(x) - \frac{1}{4}\left(\frac{dV^*}{dx}\right)^T g R^{-1} g^T \frac{dV^*}{dx}$, $V(0) = 0$

Off-line solution

HJB hard to solve. May not have smooth solution.

Dynamics must be known

Optimal Control: Linear Quadratic Regulator

System $\dot{x} = Ax + Bu$

Cost $V(x(t)) = \int_t^\infty (x^T Qx + u^T Ru)\, d\tau \qquad = x^T(t) P x(t)$

Differential equivalent is the Bellman equation

Scalar equation

$$0 = H(x, \frac{\partial V}{\partial x}, u) = \dot{V} + x^T Qx + u^T Ru = 2\left(\frac{\partial V}{\partial x}\right)^T \dot{x} + x^T Qx + u^T Ru = 2x^T P(Ax + Bu) + x^T Qx + u^T Ru$$

Given any stabilizing FB policy $u = -Kx$

The cost value is found by solving **Lyapunov equation = Bellman equation**

$$0 = (A - BK)^T P + P(A - BK) + Q + K^T RK$$

Matrix equation

Optimal Control is

$$u = -R^{-1} B^T P x = -Kx$$

Algebraic Riccati equation

$$0 = PA + A^T P + Q - PBR^{-1} B^T P$$

Full system dynamics must be known
Off-line solution

We want to find optimal control solutions

- Online in real-time
- Using adaptive control techniques
- Without knowing the full dynamics

Optimality in Biological Systems

Every living organism improves its control actions based on rewards received from the environment

The resources available to living organisms are usually meager. Nature uses optimal control.

Reinforcement Learning

1. Apply a control. Evaluate the benefit of that control.
2. Improve the control policy.

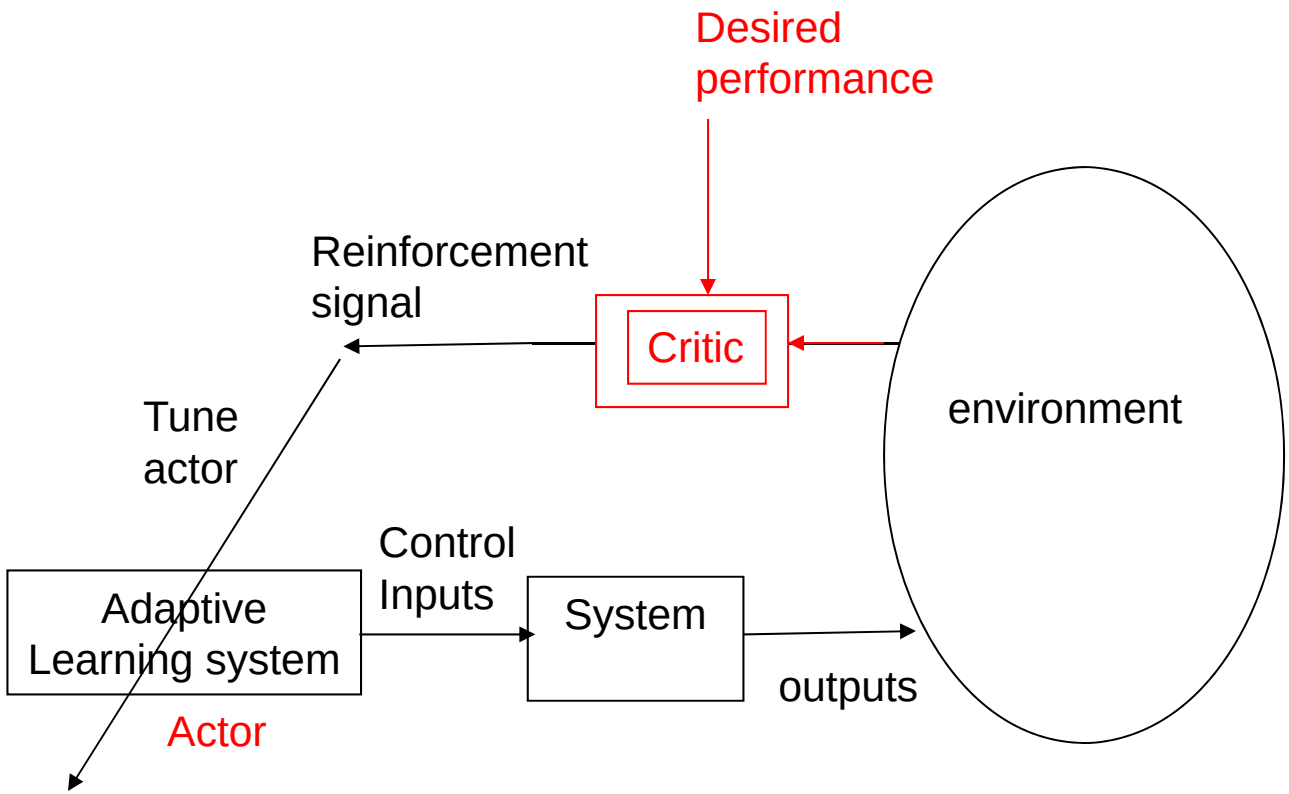
RL finds optimal policies by evaluating the effects of suboptimal policies

Different methods of learning

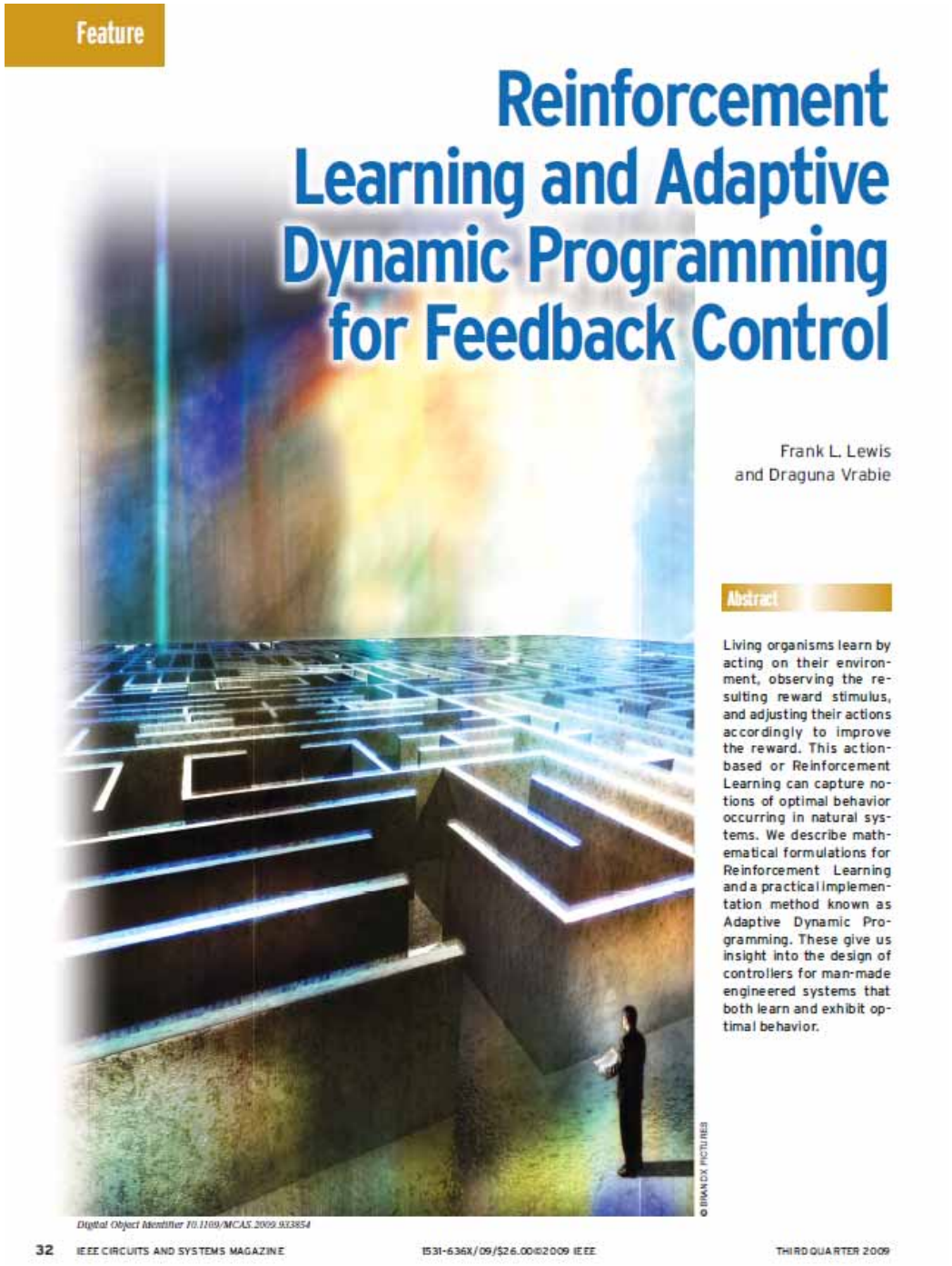
Reinforcement learning
Ivan Pavlov 1890s

We want OPTIMAL performance
- ADP- Approximate Dynamic Programming

Actor-Critic Learning



Sutton & Barto book



F.L. Lewis and D. Vrabie, “Reinforcement learning and adaptive dynamic programming for feedback control,” IEEE Circuits & Systems Magazine, Invited Feature Article, pp. 32-50, Third Quarter 2009.

IEEE Control Systems Magazine “Reinforcement learning and feedback Control,” Dec. 2012

RL has been developed for Discrete-Time Systems

Discrete-Time System Hamiltonian Function

$$H(x_k, \nabla V(x_k), h) = r(x_k, h(x_k)) + \gamma V_h(x_{k+1}) - V_h(x_k)$$

- Directly leads to temporal difference techniques
- System dynamics does not occur
- Two occurrences of value allow APPROXIMATE DYNAMIC PROGRAMMING methods

Continuous-Time System Hamiltonian Function

$$H(x, \frac{\partial V}{\partial x}, u) = \dot{V} + r(x,u) = \left(\frac{\partial V}{\partial x}\right)^T \dot{x} + r(x,u) = \left(\frac{\partial V}{\partial x}\right)^T f(x,u) + r(x,u)$$

Leads to off-line solutions if system dynamics is known
Hard to do on-line learning

- How to define temporal difference?
- System dynamics DOES occur
- Only ONE occurrence of value gradient

How can one do Policy Iteration for Unknown Continuous-Time Systems?
What is Value Iteration for Continuous-Time systems?
How can one do ADP for CT Systems?

Continuous-Time Optimal Control

To find online methods for optimal control

Focus on these two equations

System dynamics $$\dot{x} = f(x,u) = f(x) + g(x)u$$

Cost/value $$V(x(t)) = \int_t^\infty r(x,u)\,dt = \int_t^\infty (Q(x) + u^T R u)\,dt$$

Bellman Equation, in terms of the Hamiltonian function

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Stationarity condition $$\frac{\partial H}{\partial u} = 0$$

Stationary Control Policy $$u = h(x) = -\tfrac{1}{2} R^{-1} g^T(x) \frac{\partial V}{\partial x}$$

HJB equation $$0 = \left(\frac{dV^*}{dx}\right)^T f + Q(x) - \tfrac{1}{4}\left(\frac{dV^*}{dx}\right)^T g R^{-1} g^T \frac{dV^*}{dx} \quad , \quad V(0) = 0$$

Optimal Control: Linear Quadratic Regulator

System $\dot{x} = Ax + Bu$

Cost $V(x(t)) = \int_t^\infty (x^T Q x + u^T R u)\, d\tau \qquad = x^T(t) P x(t)$

Differential equivalent is the Bellman equation

$$0 = H(x, \frac{\partial V}{\partial x}, u) = \dot{V} + x^T Q x + u^T R u = 2\left(\frac{\partial V}{\partial x}\right)^T \dot{x} + x^T Q x + u^T R u = 2x^T P(Ax + Bu) + x^T Q x + u^T R u$$

Given any stabilizing FB policy $u = -Kx$

The cost value is found by solving **Lyapunov equation = Bellman equation**

$$0 = (A - BK)^T P + P(A - BK) + Q + K^T R K$$

Optimal Control is

$$u = -R^{-1} B^T P x = -Kx$$

Algebraic Riccati equation

$$0 = PA + A^T P + Q - PBR^{-1} B^T P$$

Full system dynamics must be known
Off-line solution

CT Policy Iteration – a Reinforcement Learning Technique

To avoid solving HJB equation $0 = \left(\frac{dV^*}{dx}\right)^T f + Q(x) - \frac{1}{4}\left(\frac{dV^*}{dx}\right)^T g R^{-1} g^T \frac{dV^*}{dx}$

Find cost for any given admissible *u(x)*

$$0 = \left(\frac{\partial V}{\partial x}\right)^T f(x,u) + r(x,u) \equiv H(x, \frac{\partial V}{\partial x}, u)$$

CT Bellman equation

Scalar equation

Utility $r(x,u) = Q(x) + u^T R u$

Policy Iteration Solution

Pick stabilizing initial control policy

Policy Evaluation - Find cost, Bellman eq.

$$0 = \left(\frac{\partial V_j}{\partial x}\right)^T f(x, h_j(x)) + r(x, h_j(x))$$

$$V_j(0) = 0$$

Policy improvement - Update control

$$h_{j+1}(x) = -\tfrac{1}{2} R^{-1} g^T(x) \frac{\partial V_j}{\partial x}$$

- Convergence proved by Leake and Liu 1967, Saridis 1979 if Lyapunov eq. solved exactly
- Beard & Saridis used Galerkin Integrals to solve Lyapunov eq.
- Abu Khalaf & Lewis used NN to approx. V for nonlinear systems and proved convergence

Full system dynamics must be known
Off-line solution

LQR Policy iteration = Kleinman algorithm

1. For a given control policy $u = -K_j x$ solve for the cost:

$$0 = A_j^T P_j + P_j A_j + Q + K_j^T R K_j$$

Bellman eq. = Lyapunov eq.

Matrix equation

$$A_j = A - BK_j$$

2. Improve policy:

$$K_{j+1} = R^{-1} B^T P_j$$

- If started with a stabilizing control policy K_0 the matrix P_j monotonically converges to the unique positive definite solution of the Riccati equation.
- Every iteration step will return a stabilizing controller.
- The system has to be known.

- OFF-LINE DESIGN
- MUST SOLVE LYAPUNOV EQUATION AT EACH STEP.

Kleinman 1968

Integral Reinforcement Learning

Work of Draguna Vrabie

$$\dot{x} = f(x) + g(x)u$$

Can Avoid knowledge of drift term *f(x)*

Policy iteration requires repeated solution of the CT Bellman equation

$$0 = \dot{V} + r(x, u(x)) = \left(\frac{\partial V}{\partial x}\right)^T \dot{x} + r(x, u(x)) = \left(\frac{\partial V}{\partial x}\right)^T f(x, u(x)) + Q(x) + u^T R u \equiv H(x, \frac{\partial V}{\partial x}, u(x))$$

This can be done online without knowing *f(x)*
using measurements of *x(t), u(t)* along the system trajectories

Lemma 1 – Draguna Vrabie

value $V(x(t)) = \int_t^{\infty} r(x,u)\, d\tau$

$$0 = \left(\frac{\partial V}{\partial x}\right)^T f(x,u) + r(x,u) \equiv H(x, \frac{\partial V}{\partial x}, u), \quad V(0) = 0$$

Is equivalent to Integral reinf. form for the CT Bellman eq.

$$V(x(t)) = \int_t^{t+T} r(x,u)\, d\tau \quad + \quad V(x(t+T)), \quad V(0) = 0$$

Solves Bellman equation without knowing *f(x,u)*

Proof:

$$\frac{d(V(x))}{dt} = \left(\frac{\partial V}{\partial x}\right)^T f(x,u) = -r(x,u)$$

$$\int_t^{t+T} r(x,u)\, d\tau = -\int_t^{t+T} d(V(x)) = V(x(t)) - V(x(t+T))$$

Allows definition of temporal difference error for CT systems

$$e(t) = -V(x(t)) + \int_t^{t+T} r(x,u)\, d\tau \quad + \quad V(x(t+T))$$

value $$V(x(t)) = \int_t^{\infty} r(x,u)\, d\tau$$

Lemma 1 – Draguna Vrabie

$$0 = \left(\frac{\partial V}{\partial x}\right)^T f(x,u) + r(x,u) \equiv H(x, \frac{\partial V}{\partial x}, u), \quad V(0) = 0$$

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Solves Bellman equation without knowing *f(x,u)*

Allows definition of temporal difference error for CT systems

$$e(t) = -V(x(t)) + \int_t^{t+T} r(x,u)\, d\tau \quad + \quad V(x(t+T))$$

Lemma 1 - D. Vrabie- LQR case

$$A_c^T P + P A_c + L^T R L + Q = 0$$

$$A_c = A - BL$$

is equivalent to Integral reinf. form

$$x^T(t)Px(t) = \int_t^{t+T} x^T(\tau)(Q + L^T R L)x(\tau)d\tau + x^T(t+T)Px(t+T)$$

Solves Lyapunov equation without knowing A or B

Proof:

$$\frac{d(x^T P x)}{dt} = x^T(A_c^T P + P A_c)x = -x^T(L^T R L + Q)x$$

$$\int_t^{t+T} x^T(Q + L^T R L)x d\tau = -\int_t^{t+T} d(x^T P x) = x^T(t)Px(t) - x^T(t+T)Px(t+T)$$

Lemma 1 - D. Vrabie- LQR case

$$A_c^T P + P A_c + L^T R L + Q = 0$$

Matrix equation

$$A_c = A - BL$$

is equivalent to Integral reinf. form

Scalar equation

$$x^T(t) P x(t) = \int_t^{t+T} x^T(\tau)(Q + L^T R L) x(\tau) d\tau + x^T(t+T) P x(t+T)$$

Solves Lyapunov equation without knowing A or B

Integral Reinforcement Learning (IRL)- Draguna Vrabie

IRL Policy iteration

Policy evaluation- IRL Bellman Equation

Cost update $$V_k(x(t)) = \int_t^{t+T} r(x,u_k)\,dt \quad + \quad V_k(x(t+T))$$

CT Bellman eq.

f(x) and *g(x)* do not appear

Equivalent to $$0 = \left(\frac{\partial V}{\partial x}\right)^T f(x,u) + r(x,u) \equiv H(x,\frac{\partial V}{\partial x},u)$$

Solves Bellman eq. (nonlinear Lyapunov eq.) without knowing system dynamics

Policy improvement

Control gain update $$u_{k+1} = h_{k+1}(x) = -\tfrac{1}{2} R^{-1} g^T(x) \frac{\partial V_k}{\partial x}$$

g(x) needed for control update

Initial stabilizing control is needed

Converges to solution to HJB eq. $$0 = \left(\frac{dV^*}{dx}\right)^T f + Q(x) - \tfrac{1}{4}\left(\frac{dV^*}{dx}\right)^T g R^{-1} g^T \frac{dV^*}{dx}$$

D. Vrabie proved convergence to the optimal value and control

Integral Reinforcement Learning (IRL)- Draguna Vrabie

CT LQR Case

Value function is quadratic $V(x(t)) = x^T(t) P x(t)$ $\quad u_k(t) = -L_k x(t)$

CT Bellman eq.

$$x^T(t) P_k x(t) = \int_t^{t+T} x^T(\tau)(Q + L_k{}^T R L_k) x(\tau) d\tau + x^T(t+T) P_k x(t+T)$$

Solves Lyapunov equation without knowing A or B

is equivalent to

$$A_k{}^T P_k + P_k A_k + L_k^T R L_k + Q = 0$$

$$A_k = A - B L_k$$

$$L_{k+1} = R^{-1} B^T P_k$$

Only B is needed

Converges to solution to ARE

$$0 = PA + A^T P + Q - PBR^{-1} B^T P$$

Theorem – D. Vrabie

This algorithm converges and is equivalent to Kleinman's Algorithm

This is a data-based approach that uses measurements of x(t), u(t)
Instead of the plant dynamical model.

Another View- Bellman Optimality Equation Is a Fixed Point Equation

Dimitri Bertsekas
Warren Powell
Sean Meyn
D. Vrabie

$$V^*(x(t)) = \min_{\substack{u(\tau) \\ t \le \tau < t+\Delta t}} \left\{ \int_t^{t+\Delta t} r(x(\tau), u(\tau)) d\tau \quad + \quad V^*(x(t+\Delta t)) \right\}$$

or

Lewis, Vrabie, Syrmos 2012

$$0 = \min_{\substack{u(\tau) \\ t \le \tau < t+\Delta t}} \left\{ \int_t^{t+\Delta t} r(x(\tau), u(\tau)) d\tau \quad + \quad V^*(x(t+\Delta t)) - V^*(x(t)) \right\}$$

Policy must be stabilizing to solve this eq.

Define Contraction map

Bellman Eq.
$$0 = \int_t^{t+\Delta t} r(x(\tau), u_k(\tau)) d\tau \quad + \quad V_k(x(t+\Delta t)) - V_k(x(t))$$

$$u_{k+1}(x(t)) = \arg\min_{\substack{u(\tau) \\ t \le \tau < t+\Delta t}} \left\{ \int_t^{t+\Delta t} r(x(\tau), u(\tau)) d\tau \quad + \quad V_k(x(t+\Delta t)) - V_k(x(t)) \right\}$$

equivalent

Recall
$$0 = \left(\frac{\partial V_k}{\partial x} \right)^T f(x, u_k(x)) + r(x, u_k(x))$$

CT Policy Iteration – How to implement online?
Linear Systems Quadratic Cost- LQR

Value function is quadratic $V(x(t)) = x^T(t) P x(t)$

Policy evaluation- solve IRL Bellman Equation

$$x^T(t) P_k x(t) = \int_t^{t+T} x^T(\tau)(Q + L_k{}^T R L_k) x(\tau)\, d\tau \quad + \quad x^T(t+T) P_k x(t+T)$$

$$x^T(t) P_k x(t) - x^T(t+T) P_k x(t+T) = \int_t^{t+T} x^T(\tau)(Q + L_k{}^T R L_k) x(\tau)\, d\tau$$

$$\begin{bmatrix} x^1(t) & x^2(t) \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x^1(t) \\ x^2(t) \end{bmatrix} - \begin{bmatrix} x^1(t+T) & x^2(t+T) \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x^1(t+T) \\ x^2(t+T) \end{bmatrix}$$

$$= \begin{bmatrix} p_{11} & p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} (x^1)^2 \\ 2x^1x^2 \\ (x^2)^2 \end{bmatrix}_{(t)} - \begin{bmatrix} p_{11} & p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} (x^1)^2 \\ 2x^1x^2 \\ (x^2)^2 \end{bmatrix}_{(t+T)}$$

$$= \bar{p}_k^T \left[\bar{x}(t) - \bar{x}(t+T) \right]$$

Quadratic basis set

Algorithm Implementation

Critic update

$$x^T(t)P_k x(t) = \int_t^{t+T} x^T(\tau)(Q + L_k{}^T R L_k)x(\tau)d\tau + x^T(t+T)P_k x(t+T)$$

Use Kronecker product $vec(ABC) = \left(C^T \otimes A\right)vec(B)$

To set this up as $\bar{x}(t) = x(t) \otimes x(t)$ is the quadratic basis set

$$\bar{p}_k{}^T \bar{x}(t) = \int_t^{t+T} x(\tau)^T (Q + L_k{}^T R L_k)x(\tau)d\tau + \bar{p}_k{}^T \bar{x}(t+T)$$

c.f. Linear in the parameters system ID

$$\bar{p}_k{}^T \phi(t) \equiv \bar{p}_k{}^T \left[\bar{x}(t) - \bar{x}(t+T)\right] = \int_t^{t+T} x(\tau)^T (Q + L_k{}^T R L_k)x(\tau)d\tau$$

$$\equiv \rho(t, t+T)$$ Reinforcement on time interval *[t, t+T]*

Quadratic regression vector

Same form as standard System ID problems

Solve using RLS or batch LS

Need $n(n+1)/2$ data points along the system trajectory

$$\theta^T h(u, x) = \rho(u, x)$$

Regression matrix

Unknown parameters

Nonlinear Case- Approximate Dynamic Programming

Value Function Approximation (VFA) to Solve Bellman Equation

– Paul Werbos (ADP), Dmitri Bertsekas (NDP)

$$V_k(x(t)) = \int_t^{t+T} \left(Q(x) + u_k{}^T R u_k \right) dt + V_k(x(t+T))$$

Approximate value by Weierstrass Approximator Network $V = W^T \phi(x)$

$$W_k^T \phi(x(t)) = \int_t^{t+T} \left(Q(x) + u_k{}^T R u_k \right) dt + W_k^T \phi(x(t+T))$$

$$W_k^T \underbrace{\left[\phi(x(t)) - \phi(x(t+T)) \right]}_{\text{regression vector}} = \underbrace{\int_t^{t+T} \left(Q(x) + u_k{}^T R u_k \right) dt}_{\text{Reinforcement on time interval } [t,\ t+T]}$$

Scalar equation with vector unknowns

Now use RLS along the trajectory to get new weights W_{k+1}

Then find updated FB

$$u_{k+1} = h_{k+1}(x) = -\tfrac{1}{2} R^{-1} g^T(x) \frac{\partial V_k}{\partial x} = -\tfrac{1}{2} R^{-1} g^T(x) \left[\frac{\partial \phi(x(t))}{\partial x(t)} \right]^T W_k$$

Direct Optimal Adaptive Control for Partially Unknown CT Systems

Integral Reinforcement Learning (IRL)

This is a data-based approach that uses measurements of *x(t), u(t)*
Instead of the plant dynamical model.

1. Select initial control policy
2. Find associated cost

Bellman Equation
Solves Lyapunov eq. without knowing dynamics

$$\bar{p}_k^T \left[\bar{x}(t) - \bar{x}(t+T)\right] = \int_t^{t+T} x(\tau)^T (Q + L_k^T R L_k) x(\tau) d\tau = \rho(t, t+T)$$

3. Improve control $L_{k+1} = R^{-1} B^T P_k$

observe *x(t)*

apply $u^k(t) = L_k x(t)$

observe cost integral

observe *x(t+T)*

update P

Data set at time *[t,t+T)*

$$\left(x(t), \rho(t, t+T), x(t+T)\right)$$

A is not needed anywhere

t t+T

update control gain to L_{k+1}

do RLS until convergence to P_k

Batch LS Algorithm Implementation

Or use Recursive Least-Squares solution along the trajectory

The Critic update

$$x^T(t)P_k x(t) = \int_t^{t+T} x^T(\tau)(Q + L_k{}^T R L_k)x(\tau)d\tau + x^T(t+T)P_k x(t+T)$$

can be setup as

$$\bar{p}_k{}^T \varphi(t) \equiv \bar{p}_k{}^T \left[\bar{x}(t) - \bar{x}(t+T)\right] = \int_t^{t+T} x(\tau)^T (Q + L_k{}^T R L_k)x(\tau)d\tau \equiv d(\bar{x}(t), L_k)$$

$\bar{x}(t) = x(t) \otimes x(t)$ is the quadratic basis set

Evaluating $d(\bar{x}(t), L_k)$ for N=n(n+1)/2 trajectory points, one can setup a least squares problem to solve

$$\bar{p}_k = (XX^T)^{-1} XY$$

$$X = [\varphi(t) \quad \varphi(t+T) \quad \ldots \quad \varphi(t+NT)]$$

$$Y = [d(\bar{x}(t), K_i) \quad d(\bar{x}(t+T), K_i) \quad \ldots \quad d(\bar{x}(t+NT), K_i)]^T$$

Persistence of Excitation

$$\bar{p}_k{}^T \phi(t) \equiv \quad \bar{p}_k{}^T \left[\bar{x}(t) - \bar{x}(t+T)\right] = \int_t^{t+T} x(\tau)^T (Q + L_k{}^T R L_k) x(\tau) d\tau$$

Regression vector must be PE

Relates to choice of reinforcement interval T

Implementation

Policy evaluation
Need to solve online

$$\bar{p}_k{}^T \left[\bar{x}(t) - \bar{x}(t+T)\right] = \int_t^{t+T} x(\tau)^T (Q + L_k{}^T R L_k) x(\tau) d\tau = \rho(t, t+T)$$

Add a new state= Integral Reinforcement

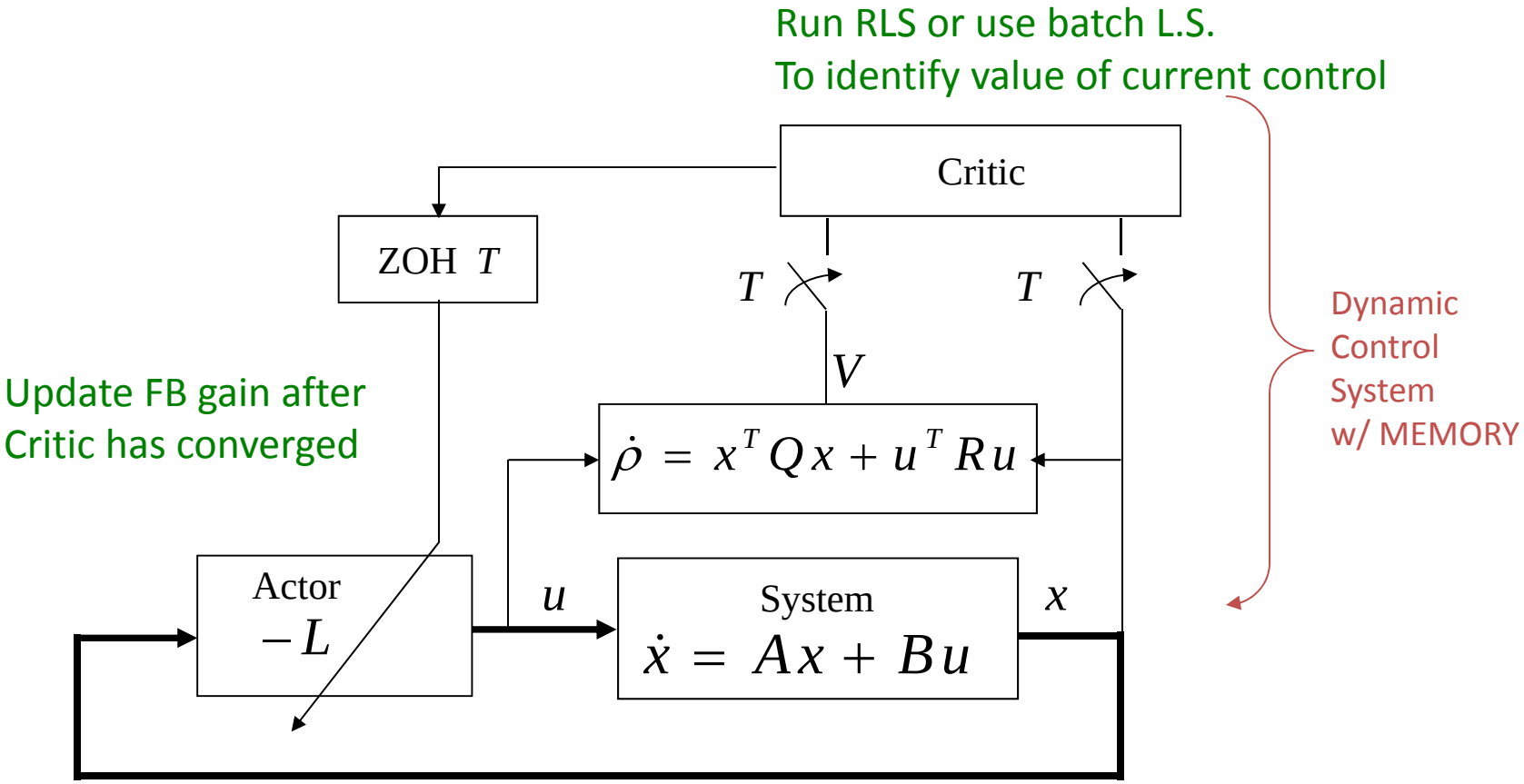
$$\dot{\rho} = x^T Q x + u^T R u$$

This is the controller dynamics or memory

Draguna Vrabie

Direct Optimal Adaptive Controller

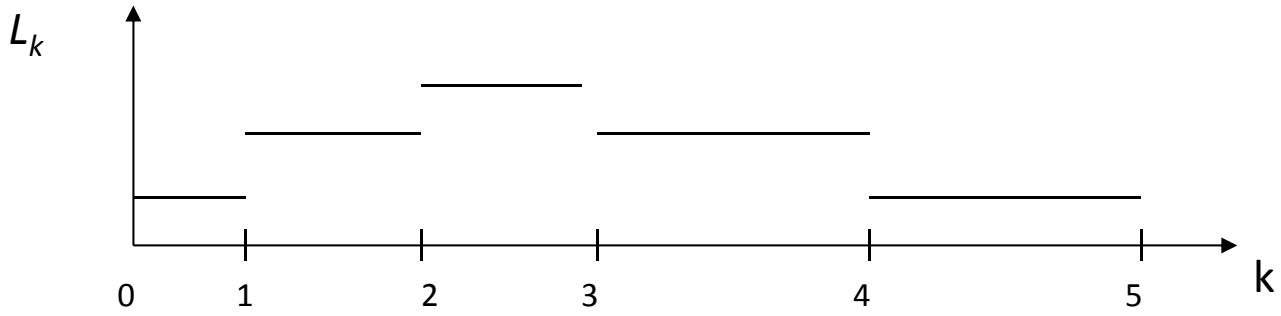
Solves Riccati Equation Online without knowing A matrix



A hybrid continuous/discrete dynamic controller
whose internal state is the observed cost over the interval

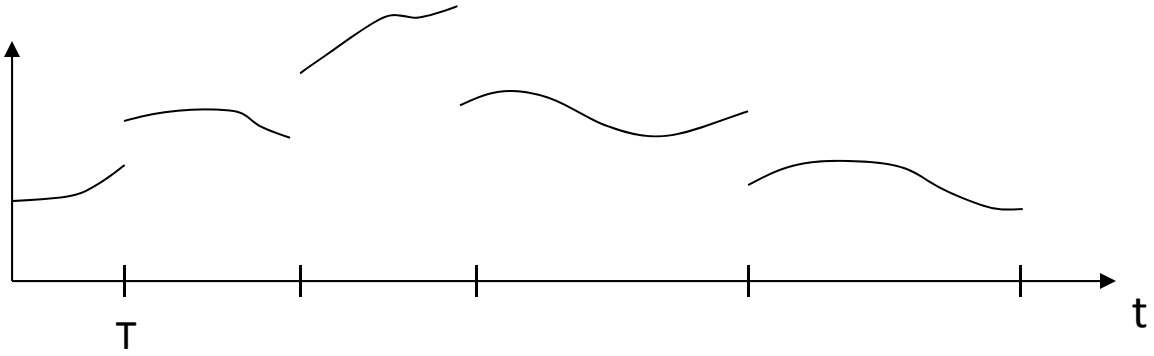
Reinforcement interval T can be selected on line on the fly – can change

Gain update (Policy)



Control

$$u_k(t) = -L_k x(t)$$



Reinforcement Intervals T need not be the same
They can be selected on-line in real time

Continuous-time control with discrete gain updates

Simulation 1- F-16 aircraft pitch rate controller

$$\dot{x} = \begin{bmatrix} -1.01887 & 0.90506 & -0.00215 \\ 0.82225 & -1.07741 & -0.17555 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Stevens and Lewis 2003

$x = [\alpha \quad q \quad \delta_e]$

$Q = I, \quad R = I$

ARE $\quad 0 = PA + A^T P + Q - PBR^{-1}B^T P$

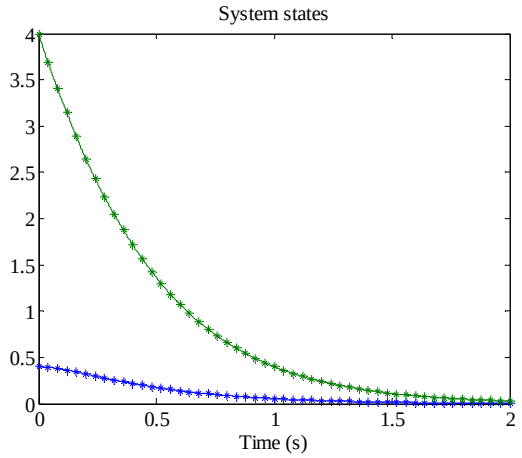
Select quadratic NN basis set for VFA

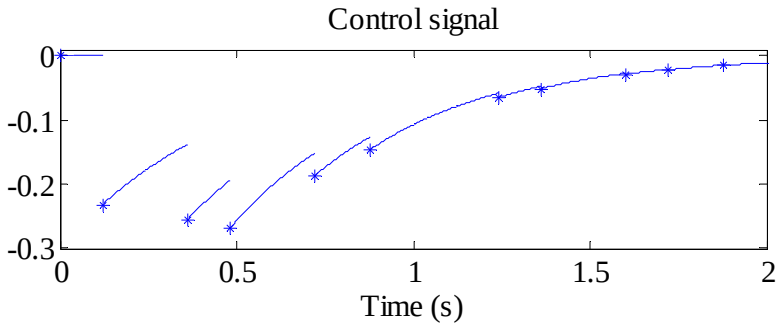
Exact solution $\quad W_1^* = [p_{11} \quad 2p_{12} \quad 2p_{13} \quad p_{22} \quad 2p_{23} \quad p_{33}]^T$

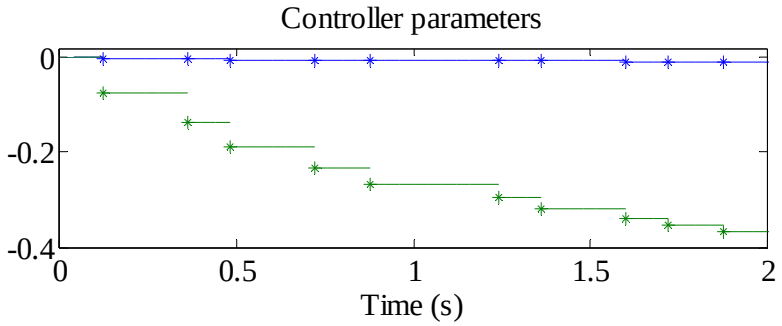
$= [1.4245 \quad 1.1682 \quad -0.1352 \quad 1.4349 \quad -0.1501 \quad 0.4329]^T$

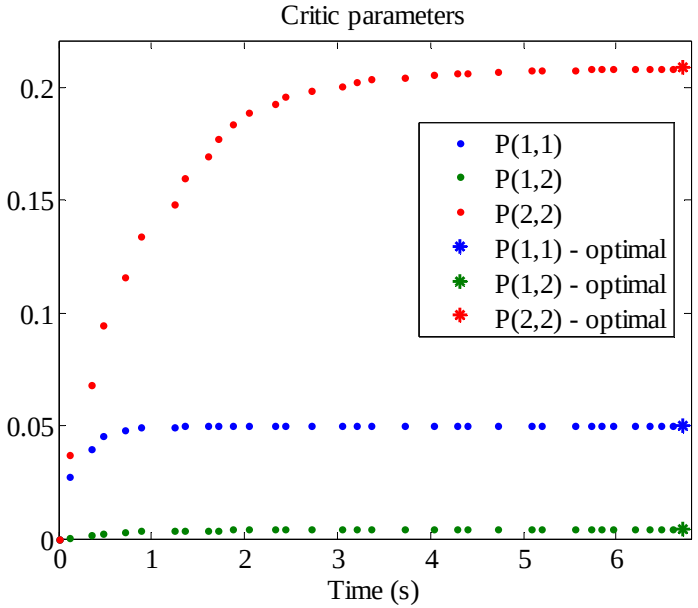
Simulations on: F-16 autopilot

A matrix not needed









Converge to SS Riccati equation soln

Solves ARE online without knowing A

$$0 = PA + A^T P + Q - PBR^{-1}B^T P$$

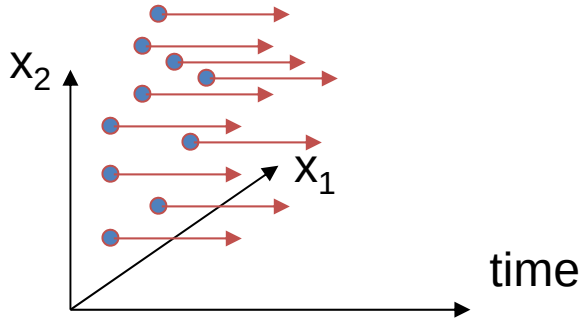
Issues with Nonlinear ADP

Selection of NN Training Set

LS local smooth solution for Critic NN update

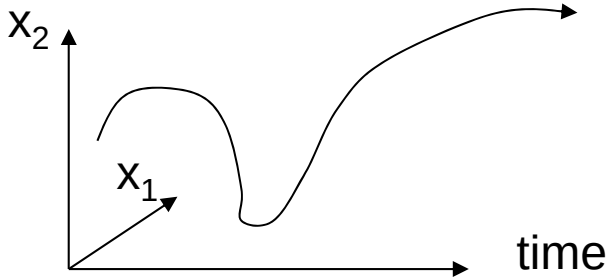
$$0 = \left(\frac{\partial V}{\partial x}\right)^T f(x,u) + r(x,u) \equiv H(x, \frac{\partial V}{\partial x}, u), \quad V(0) = 0$$

$$V(x(t)) = \int_t^{t+T} r(x,u)\, d\tau \quad + \quad V(x(t+T)), \quad V(0) = 0$$



Integral over a region of state-space
Approximate using a set of points

Batch LS



Take sample points along a single trajectory

Recursive Least-Squares RLS

Set of points over a region vs. points along a trajectory

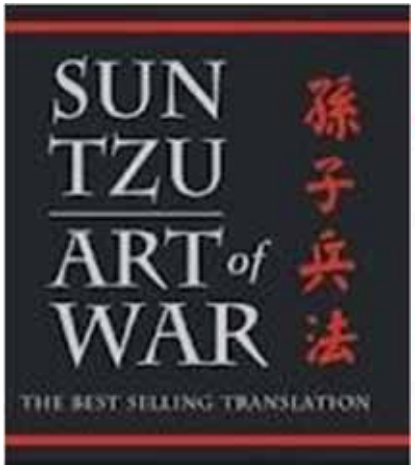
For Linear systems- these are the same

For Nonlinear systems

Persistence of excitation is needed to solve for the weights

But EXPLORATION is needed to identify the complete value function

- PE Versus Exploration



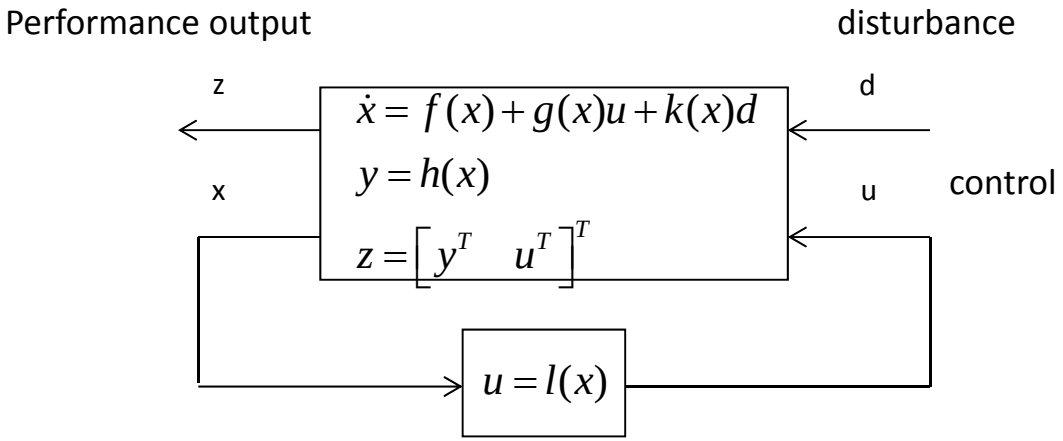
孙子兵法

500 BC

Sun Tz bin fa

2. H-Infinity Control Using Neural Networks

System



L_2 Gain Problem

Find control *u(t)* so that

$$\frac{\int_0^\infty \|z(t)\|^2 dt}{\int_0^\infty \|d(t)\|^2 dt} = \frac{\int_0^\infty (h^T h + \|u\|^2) dt}{\int_0^\infty \|d(t)\|^2 dt} \le \gamma^2$$

For all L_2 disturbances
And a prescribed gain γ^2

Zero-Sum differential game

Nature as the opposing player

2. Online Zero-Sum Differential Games

H-infinity Control

System $\dot{x} = f(x,u) = f(x) + g(x)u + k(x)d$

$y = h(x)$

2 players

Cost $V(x(t),u,d) = \int_t^\infty \left(h^T h + u^T R u - \gamma^2 \|d\|^2 \right) dt \equiv \int_t^\infty r(x,u,d)\, dt$

Leibniz gives
Differential equivalent

Differential equivalent is ZS game Bellman equation

$$0 = r(x,u,d) + \dot{V} = r(x,u,d) + \left(\nabla V\right)^T \left(f(x) + g(x)u + k(x)d \right) \equiv H(x, \frac{\partial V}{\partial x}, u, d)$$

$$V(0) = 0$$

Given any stabilizing control and disturbance policies $u(x), d(x)$

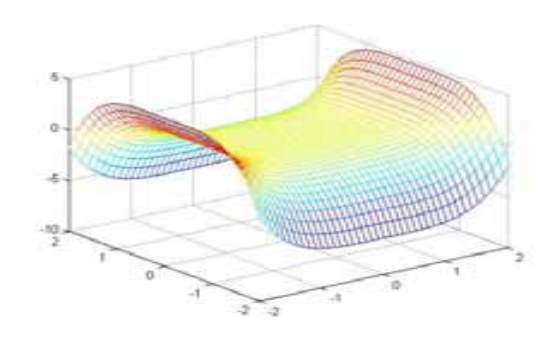
the cost value is found by solving this nonlinear Lyapunov equation

Define 2-player zero-sum game as

$$V^*(x(0)) = \min_u \max_d V(x(0),u,d) \quad = \min_u \max_d \int_0^\infty \left(h^T(x)h(x) + u^T R u - \gamma^2 \|d\|^2 \right) dt$$

The game has a unique value (saddle-point solution) iff the Nash condition holds

$$\min_u \max_d V(x(0),u,d) = \max_d \min_u V(x(0),u,d)$$



A necessary condition for this is the Isaacs Condition

$$\min_u \max_d H(x,\nabla V,u,d) = \max_d \min_u H(x,\nabla V,u,d)$$

Stationarity Conditions

$$0 = \frac{\partial H}{\partial u},\ 0 = \frac{\partial H}{\partial d}$$

Game saddle point solution found from Hamiltonian - BELLMAN EQUATION

$$H(x,\frac{\partial V}{\partial x},u,d)=h^T h+u^T Ru-\gamma^2\left\|d\right\|^2+\left(\nabla V\right)^T\left(f(x)+g(x)u+k(x)d\right)$$

Optimal control/dist. policies found by stationarity conditions $0=\frac{\partial H}{\partial u},\ 0=\frac{\partial H}{\partial d}$

$$u=-\frac{1}{2}R^{-1}g^T(x)\nabla V$$

$$d=\frac{1}{2\gamma^2}k^T(x)\nabla V$$

HJI equation

$$0=H(x,\nabla V,u^*,d^*)$$

$$=h^T h+\nabla V^T(x)f(x)-\frac{1}{4}\nabla V^T(x)g(x)R^{-1}g^T(x)\nabla V(x)+\frac{1}{4\gamma^2}\nabla V^T(x)kk^T\nabla V(x)$$

$$V(0)=0$$

('Nonlinear Game Riccati' equation)

Linear Quadratic Zero-Sum Games

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2$$

$$y = Cx$$

$$-J_2(x(t), u_1, u_2) = J_1(x(t), u_1, u_2) = \frac{1}{2}\int_t^{\infty} (x^T Q x + u_1^T R_{11} u_1 - u_2^T R_{12} u_2)\, d\tau \qquad , \quad Q = C^T C$$

Game Algebraic Riccati Equation

$$0 = A^T P + PA + Q - PB_1 {R_{11}}^{-1} B_1^T P + PB_2 {R_{12}}^{-1} B_2^T P$$

$$u_1 = -K_1 x \equiv -R_{11}^{-1} B_1^T P x, \qquad u_2 = K_2 x \equiv R_{12}^{-1} B_2^T P x$$

Policy Iteration Algorithm to Solve HJI

Start with stabilizing initial control policy $u_0(x)$

1. For a given control policy $u_j(x)$ solve for the value $V_{j+1}(x(t))$

$$0 = h^T h + \nabla V_{j+1}^T(x)\left(f(x) + g(x)u_j(x)\right) + u_j^T(x)Ru_j(x) + \frac{1}{4\gamma^2}\nabla V_{j+1}^T(x)kk^T\nabla V_{j+1}(x)$$

$$V_{j+1}(0) = 0$$

HJ equation
Nonlinear 'Riccati' eq.
Solve for Available Storage

2. Improve policy:

$$u_{j+1}(x) = -\frac{1}{2}R^{-1}g^T(x)\nabla V_{j+1}$$

Minimal nnd solution of HJ equation is the Available Storage for $u_j(x)$

Off-line solution
Nonlinear HJ equation must be solved at each step

Double Policy Iteration Algorithm to Solve HJI

Add inner loop to solve for available storage

Start with stabilizing initial policy $u_0(x)$

1. For a given control policy $u_j(x)$ solve for the value $V_{j+1}(x(t))$

2. Set $d^0 = 0$. For i=0,1,… solve for $V_j^i(x(t)), d^{i+1}$

$$0 = h^T h + \nabla V_j^{iT}(x)(f + gu_j + kd^i) + u_j{}^T R u_j - \gamma^2 \left\| d^i \right\|^2$$

ZS game Bellman eq.

$$d^{i+1} = \frac{1}{2\gamma^2} k^T(x) \nabla V_j^i$$

On convergence set $V_{j+1}(x) = V_j^i(x)$

3. Improve policy:

$$u_{j+1}(x) = -\frac{1}{2} R^{-1} g^T(x) \nabla V_{j+1}$$

- Convergence proved by Van der Schaft if can solve nonlinear Lyapunov equation exactly
- Abu Khalaf & Lewis used NN to approximate V for nonlinear systems and proved convergence

Off-line solution

Nonlinear Lyapunov equation must be solved at each step

Draguna Vrabie

Online Learning for Two-Player Zero-Sum Games

Online game solutions without knowing *A* matrix

- System dynamics

$$\dot{x} = Ax + B_1 w + B_2 u, \quad x \in R^n, \ u \in R^m, \ w \in R^q$$

- Cost function

$$V(x_0, u, w) = \int_0^\infty (x^T C^T C x + u^T u - w^T w) dt$$

- Goal: saddle point

$$V(x_0, \tilde{u}, w^*) \geq V(x_0, u^*, w^*) \geq V(x_0, u^*, \tilde{w})$$

- State-feedback stabilizing solution

$$u^* = -B_2{}^T P x, \quad w^* = B_1{}^T P x, \quad V(x_0, u^*, w^*) = x_0{}^T P x_0$$

$$0 = A^T P + PA + C^T C - P(B_2 B_2{}^T - B_1 B_1{}^T) P$$

Online Policy Iteration for 2-player ZS games

Options:

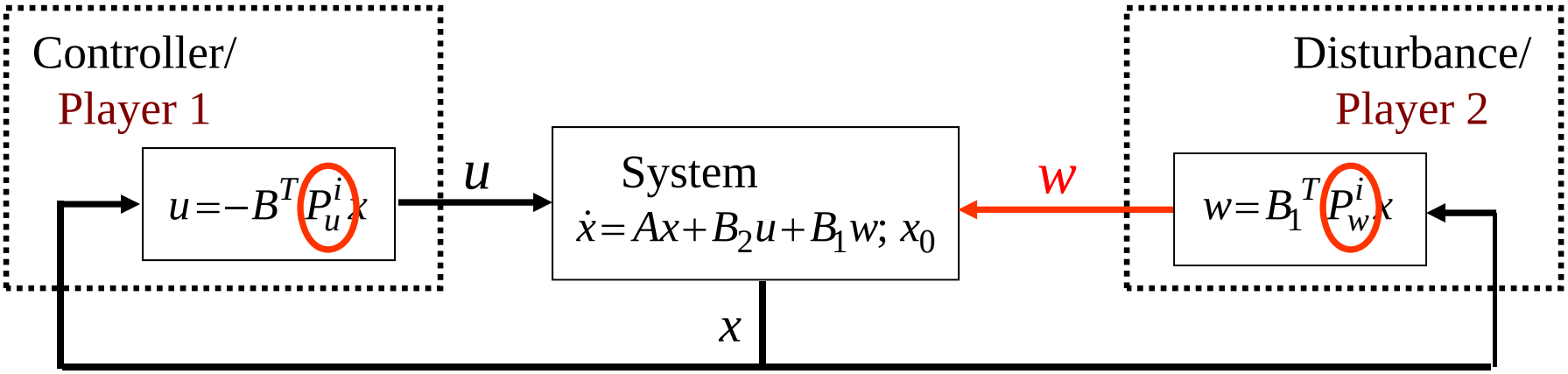
1. Both players learn online (two critics) to optimize their behavior policies

a) simultaneously

b) taking turns – while one is learning the other player maintains a fixed policy

2. Only one player learns online => single critic

- the other player uses a fixed policy and only updates it at discrete moments based on information on the policy of his opponent



Parameters that define the policies of the players

Online Nash equilibrium Learning

The game is played as follows:

1. The game starts while Player 2 (the disturbance) does not play.
2. Player 1
    a. plays the game without opponent and
    b. uses reinforcement learning to find the optimal behavior which minimizes its value;
    c. then informs Player 2 on his new optimized value fn.
3. Player 2 starts playing using the value fn. of his opponent.
4. Player 1
    a. corrects iteratively his own behavior using reinforcement learning such that its value is again minimized;
    b. then informs Player 2 on his new optimized value fn.
5. Go to step 3 until the two policies are characterized by the same parameter values.

Policy Iteration for Online Zero-Sum Games

Convergence proven by Lanzon, Feng, Anderson 2009

The game is played as follows:

1. $i=1; \quad P_u^{i-1}=P_u^0=0; \qquad w_1=B_1{}^T P_u^0 x=0$

Draguna Vrabie

2. Player 1 solves online, using HDP, the Riccati equation

$$P_u^1 A + A^T P_u^1 - P_u^1 B_2 B_2{}^T P_u^1 + C^T C = 0$$

$$u_1 = -B_2{}^T P_u^1 x$$

then informs Player 2 on P_u^1

3. Player 2 uses the value P_u^i of Player 1. Computes his policy $w_i = B_1{}^T P_u^i x$

4. Player 1 solves online, using HDP, the Riccati equation;

$$Z_u^i A_u^{i-1} + A_u^{i-1T} Z_u^i - Z_u^i B_2 B_2{}^T Z_u^i + Z_u^{i-1} B_1 B_1{}^T Z_u^{i-1} = 0$$

$$P_u^i = Z_u^i + P_u^{i-1}$$

correction of the policy of Player 1

$$u_i = -B_2{}^T P_u^i x$$

then informs Player 2 on P_u^i

5. Set $i=i+1$. Go to step 3 until the two policies are characterized by the same parameter values.

Riccati equations can be solved using HDP without knowledge of the A matrix

Integral Reinforcement Learning (IRL) to solve ARE- Draguna Vrabie

CT Bellman eq.

$$x^T(t)P_k x(t) = \int_t^{t+T} x^T(\tau)(Q + L_k{}^T R L_k)x(\tau)d\tau + x^T(t+T)P_k x(t+T)$$

Solves Lyapunov equation without knowing A or B

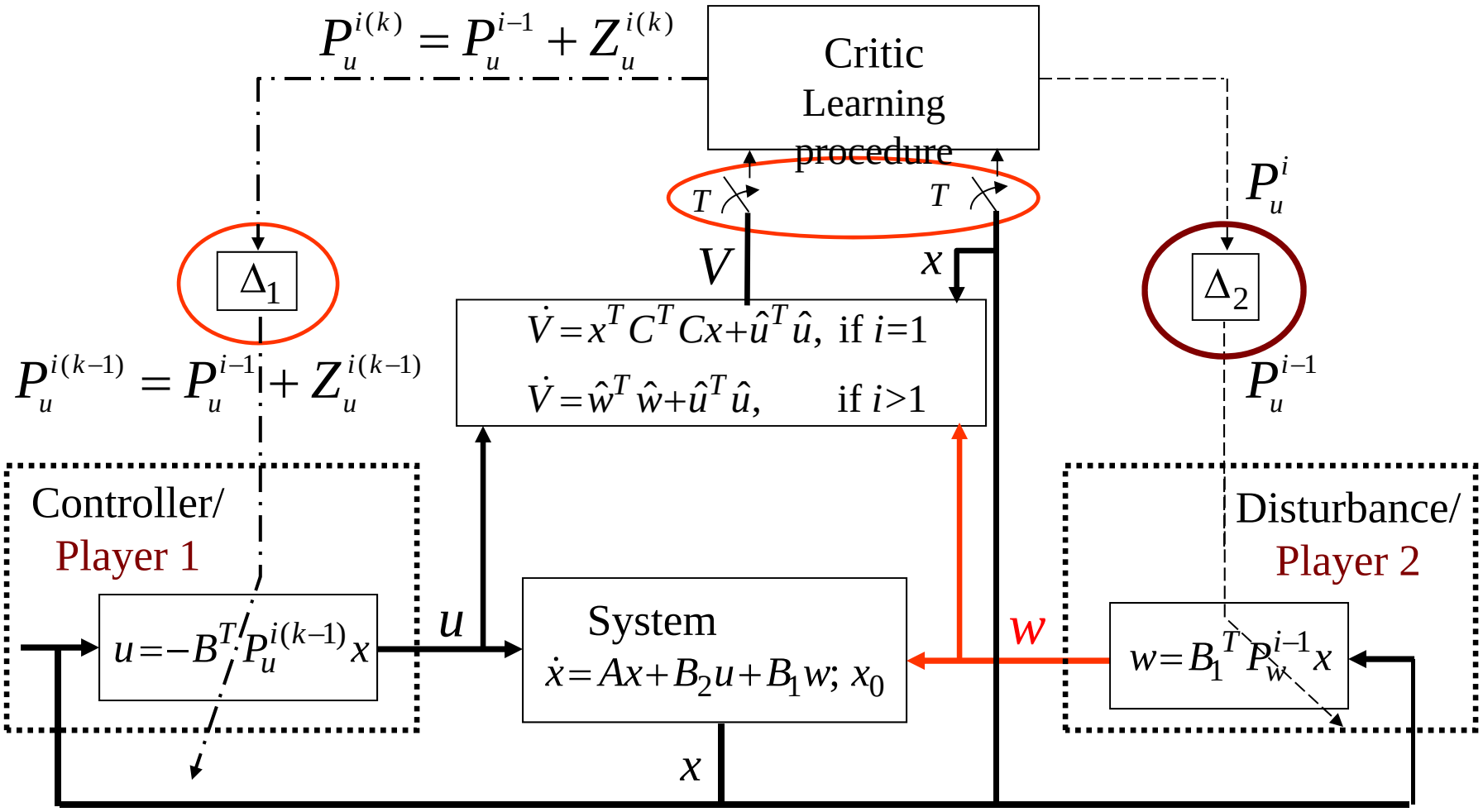
$$L_{k+1} = R^{-1}B^T P_k$$

Only B is needed

Converges to solution to ARE $\quad 0 = PA + A^T P + Q - PBR^{-1}B^T P$

This is a data-based approach that uses measurements of x(t), u(t)
Instead of the plant dynamical model.

Actor-Critic structure - three time scales



Comparison of CT IRL ADP to Discrete-Time ADP

$$\dot{x} = Ax + Bu$$

$$x(t) = [\Delta f(t) \quad \Delta P_g(t) \quad \Delta X_g(t) \quad \Delta E(t)]^T$$

$$A = \begin{bmatrix} -1/T_p & K_p/T_p & 0 & 0 \\ 0 & -1/T_T & 1/T_T & 0 \\ -1/RT_G & 0 & -1/T_G & -1/T_G \\ K_E & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1/T_G \\ 0 \end{bmatrix}$$

Frequency
Generator output
Governor position
Integral control

a. Use discrete-time ADP

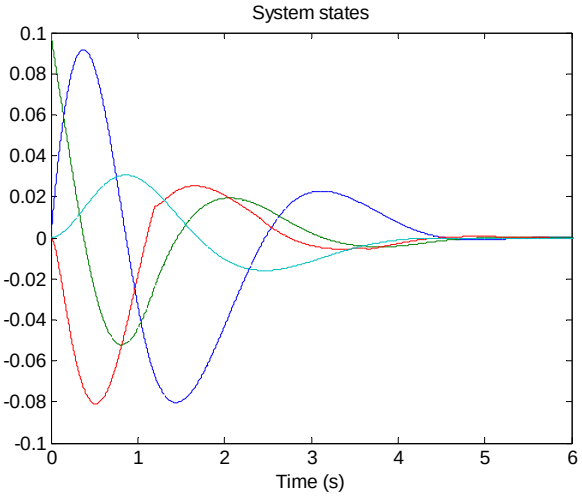
$$A^T PA - P + Q - A^T PB(B^T PB + R)^{-1} B^T PA = 0.$$

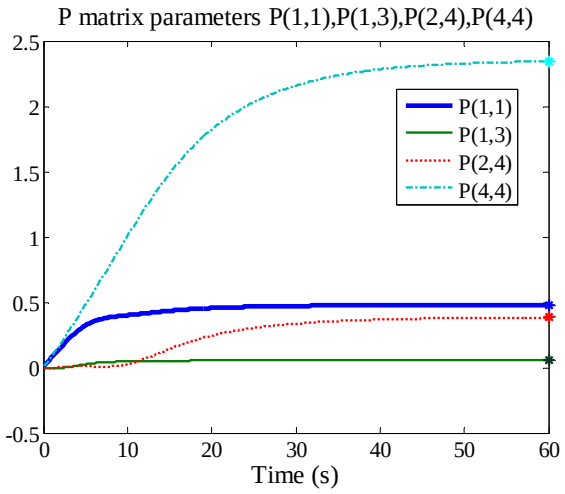
$$A = \begin{bmatrix} -0.0665 & 8 & 0 & 0 \\ 0 & -3.663 & 3.663 & 0 \\ -6.86 & 0 & -13.736 & -13.736 \\ 0.6 & 0 & 0 & 0 \end{bmatrix}, \quad B = [0 \quad 0 \quad 13.7355 \quad 0].$$

sampling period of T= 0.01s.

$$P_{DARE} = \begin{bmatrix} 0.4750 & 0.4766 & 0.0601 & 0.4751 \\ 0.4766 & 0.7831 & 0.1237 & 0.3829 \\ 0.0601 & 0.1237 & 0.0513 & 0.0298 \\ 0.4751 & 0.3829 & 0.0298 & 2.3370 \end{bmatrix}.$$

$$P_{critic\ NN} = \begin{bmatrix} 0.4802 & 0.4768 & 0.0603 & 0.4754 \\ 0.4768 & 0.7887 & 0.1239 & 0.3834 \\ 0.0603 & 0.1239 & 0.0567 & 0.0300 \\ 0.4754 & 0.3843 & 0.0300 & 2.3433 \end{bmatrix}.$$



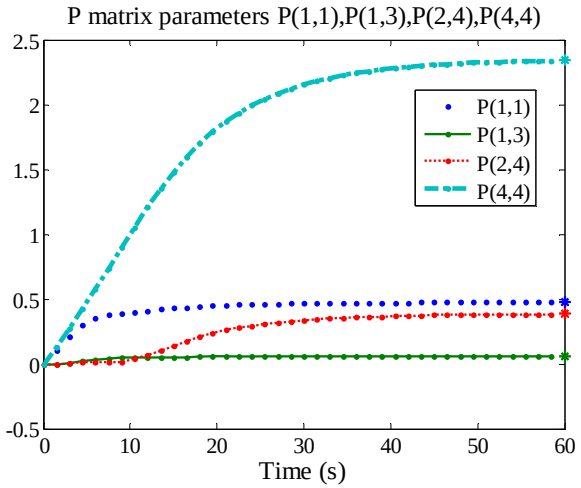


b. Use continuous-time IRL ADP

$$0 = PA + A^T P + Q - PBR^{-1}B^T P$$

Solves ARE online without knowing A

$$P_{ARE} = \begin{bmatrix} 0.4750 & 0.4766 & 0.0601 & 0.4751 \\ 0.4766 & 0.7831 & 0.1237 & 0.3829 \\ 0.0601 & 0.1237 & 0.0513 & 0.0298 \\ 0.4751 & 0.3829 & 0.0298 & 2.3370 \end{bmatrix}.$$



IRL period of T= 0.1s.

Fifteen data points $\left(x(t), x(t+T), \rho(t:t+T)\right)$

Hence, the value estimate was updated every 1.5s.

Less computation is needed using CT IRL

In DT ADP sampling period is 0.01s and the critic parameter estimates were updated every 0.15s.

Yet, the parameter estimates for the P matrix entries almost overlay each other.

Simulation- H-inf control for Electric Power Plant- LFC

$$\dot{x} = Ax + B_2 u + B_1 d$$

$$x(t) = [\Delta f(t) \quad \Delta P_g(t) \quad \Delta X_g(t) \quad \Delta E(t)]^T$$

Frequency
Generator output
Governor position
Integral control

$$A = \begin{bmatrix} -1/T_p & K_p/T_p & 0 & 0 \\ 0 & -1/T_T & 1/T_T & 0 \\ -1/RT_G & 0 & -1/T_G & -1/T_G \\ K_E & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1/T_G \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -0.0665 & 8 & 0 & 0 \\ 0 & -3.663 & 3.663 & 0 \\ -6.86 & 0 & -13.736 & -13.736 \\ 0.6 & 0 & 0 & 0 \end{bmatrix}, \quad B = [0 \quad 0 \quad 13.7355 \quad 0]^T .$$

$$B_1 = [-8 \quad 0 \quad 0 \quad 0]^T$$

Load disturbance

$$0 = A^T P + PA + C^T C - P(B_2 B_2^T - B_1 B_1^T)P$$

A is unknown
B_1, B_2 are known

Simulation result – Electric Power Plant LFC

- System – Power plant - internally stable system;
  - system state $x=[\Delta f(t) \quad \Delta P_g(t) \quad \Delta X_g(t) \quad \Delta E(t)]$ (incremental changes of: frequency deviation, generator output, governor position and integral control)
  - Player 1 - controller ; Player 2 – load disturbance
- Nash equilibrium solution

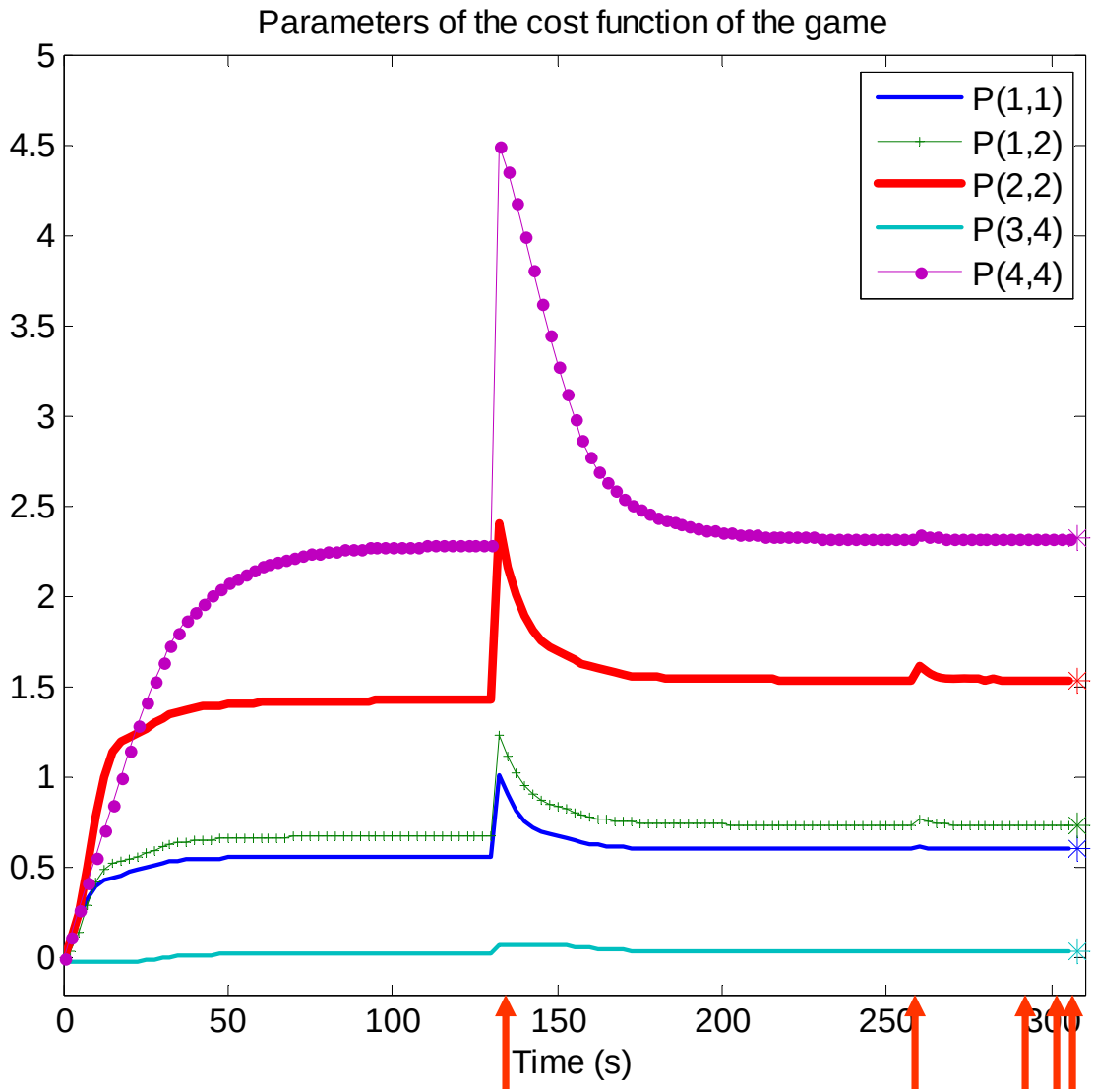
$$P_u^{\infty} = \Pi = \begin{bmatrix} 0.6036 & 0.7398 & 0.0609 & 0.5877 \\ 0.7398 & 1.5438 & 0.1702 & 0.5978 \\ 0.0609 & 0.1702 & 0.0502 & 0.0357 \\ 0.5877 & 0.5978 & 0.0357 & 2.3307 \end{bmatrix}$$

- Online learned solution using ADP – after 5 updates of the parameters of Player 2

$$P_u^5 = \begin{bmatrix} 0.6036 & 0.7399 & 0.0609 & 0.5877 \\ 0.7399 & 1.5440 & 0.1702 & 0.5979 \\ 0.0609 & 0.1702 & 0.0502 & 0.0357 \\ 0.5877 & 0.5979 & 0.0357 & 2.3307 \end{bmatrix}$$

Solves GARE online without knowing *A* $0 = A^T P + PA + C^T C - P(B_2 B_2^T - B_1 B_1^T)P$

Parameters of the critic – ARE Solution elements



- Cost function learning using least squares
- Sampling integration time T=0.1 s
- The policy of Player 1 is updated every 2.5 s
- The policy of Player 2 is updated only when the policy of Player 1 has converged
- Number of updates of Player 1 before an update of Player 2

moments when Player 2 is updated

Oscillation is a fundamental property of neural tissue

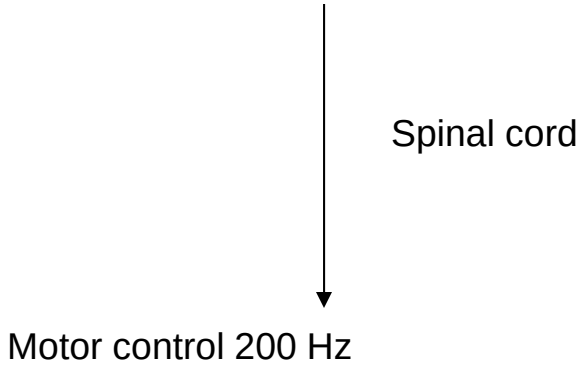
Brain has multiple adaptive clocks with different timescales

gamma rhythms 30-100 Hz, hippocampus and neocortex
high cognitive activity.

- consolidation of memory
- spatial mapping of the environment – place cells

The high frequency processing is due to the large amounts of sensorial data to be processed

theta rhythm, Hippocampus, Thalamus, 4-10 Hz
sensory processing, memory and voluntary control of movement.



D. Vrabie, F. Lewis, and Dan Levine- RL for Continuous-Time Systems

Limbic system

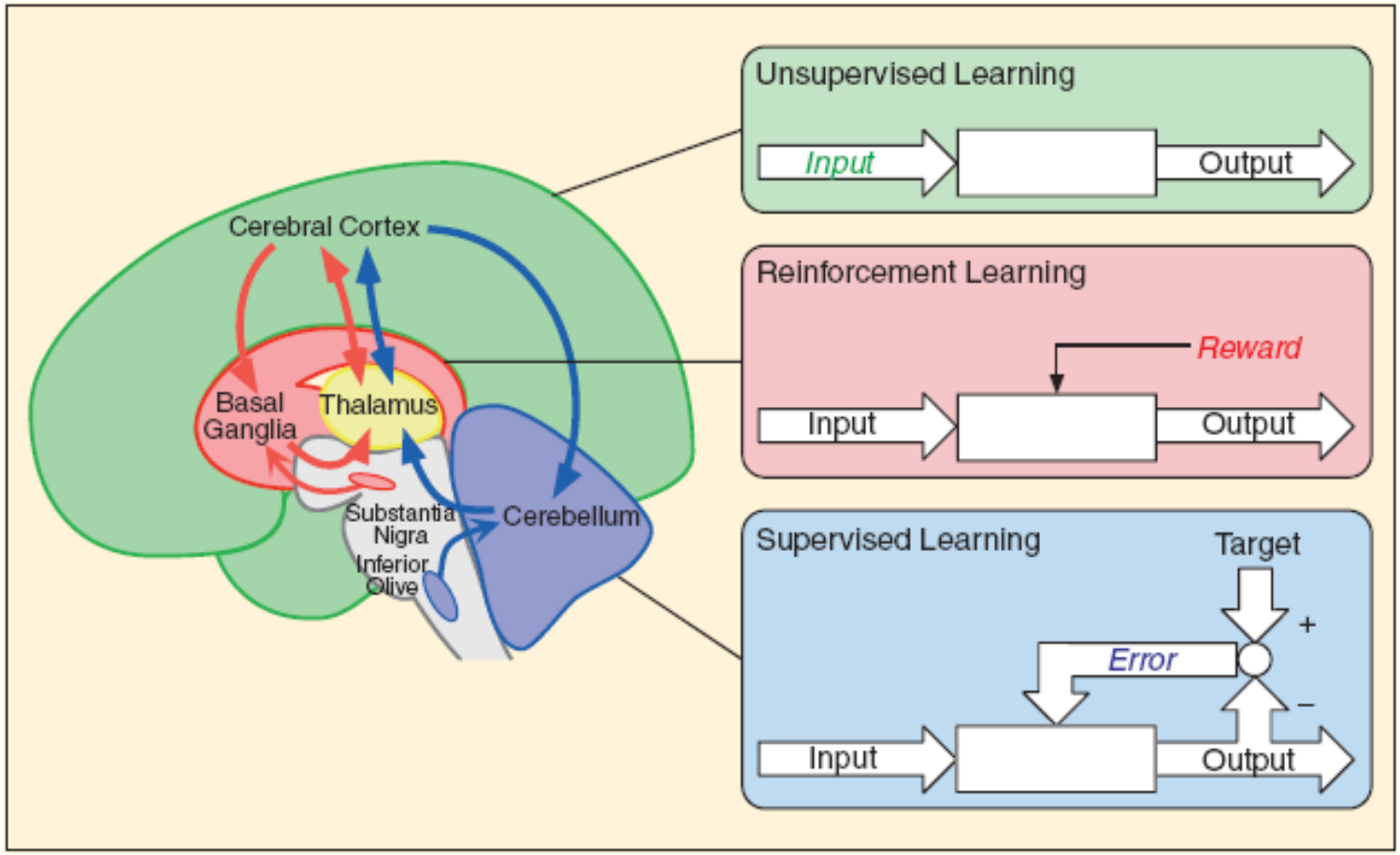
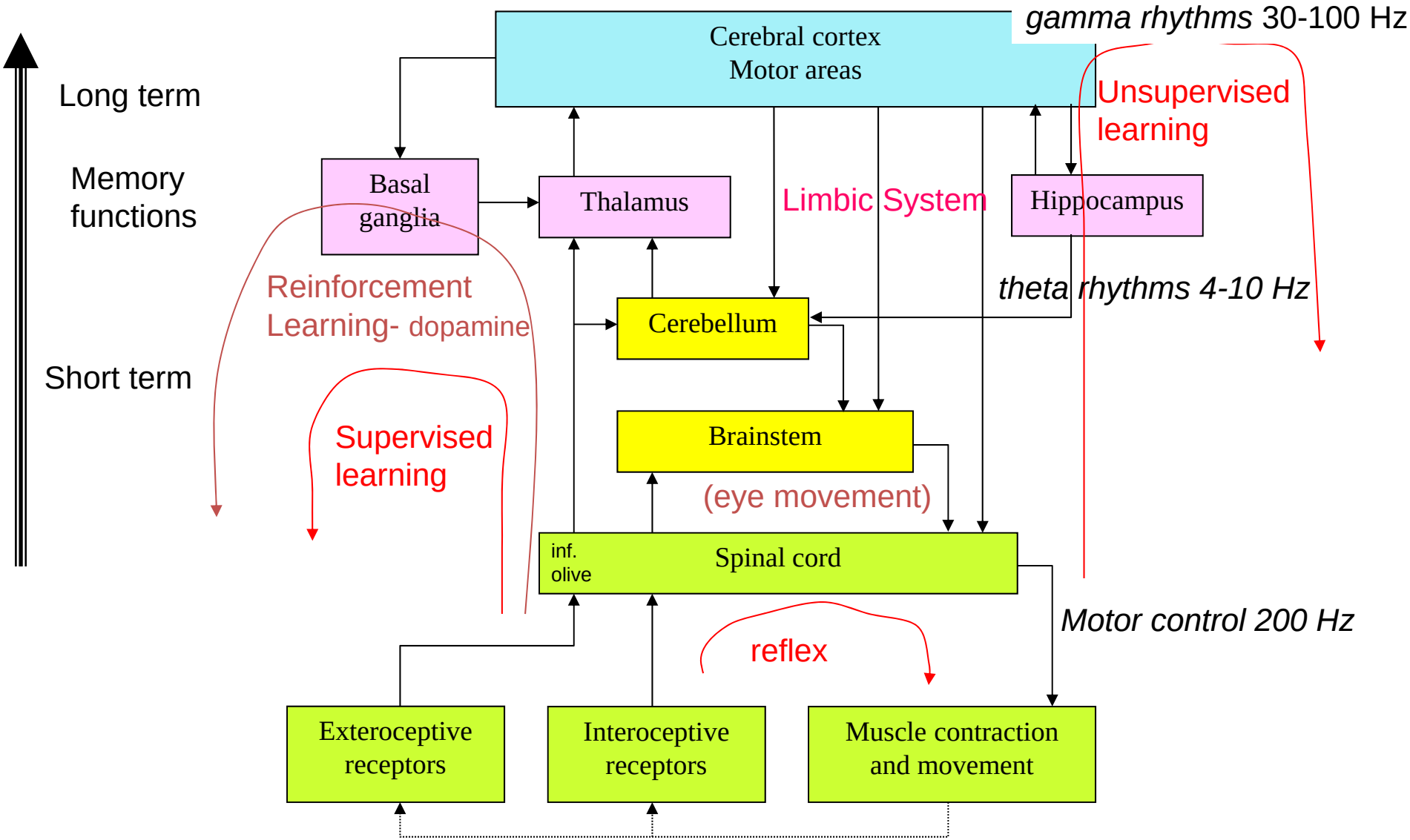


Figure 1. *Learning-oriented specialization of the cerebellum, the basal ganglia, and the cerebral cortex [1], [2]. The cerebellum is specialized for supervised learning based on the error signal encoded in the climbing fibers from the inferior olive. The basal ganglia are specialized for reinforcement learning based on the reward signal encoded in the dopaminergic fibers from the substantia nigra. The cerebral cortex is specialized for unsupervised learning based on the statistical properties of the input signal.*

Doya, Kimura, Kawato 2001

Summary of Motor Control in the Human Nervous System

picture by E. Stingu
D. Vrabie

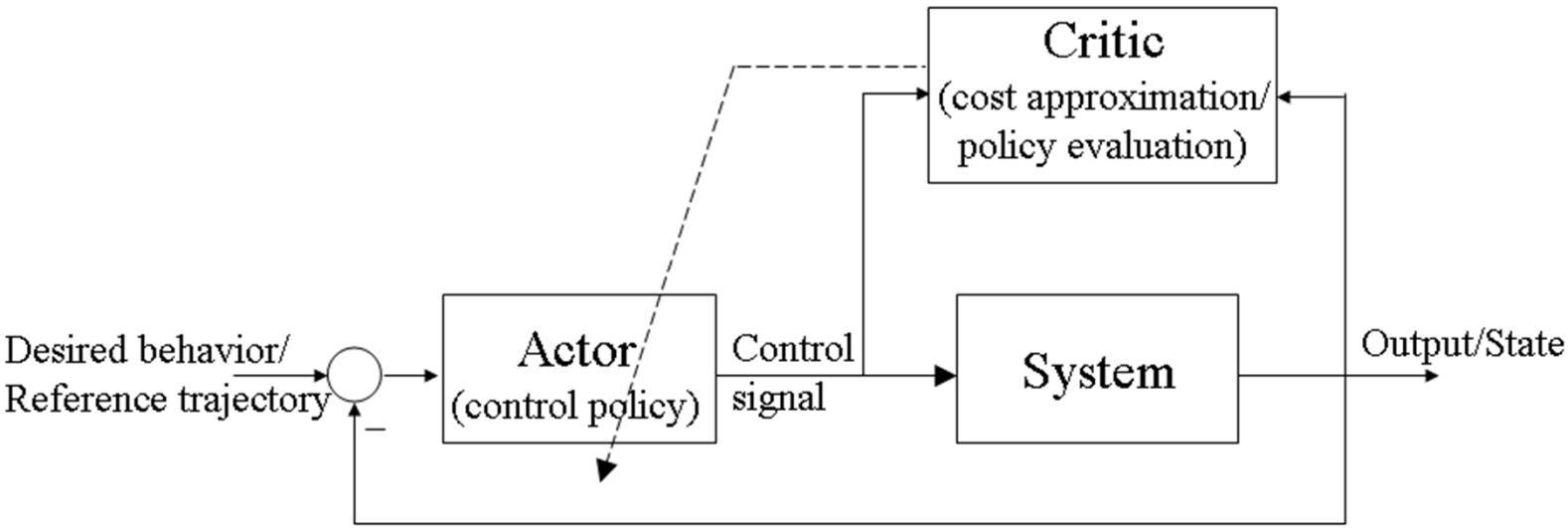


Hierarchy of multiple parallel loops

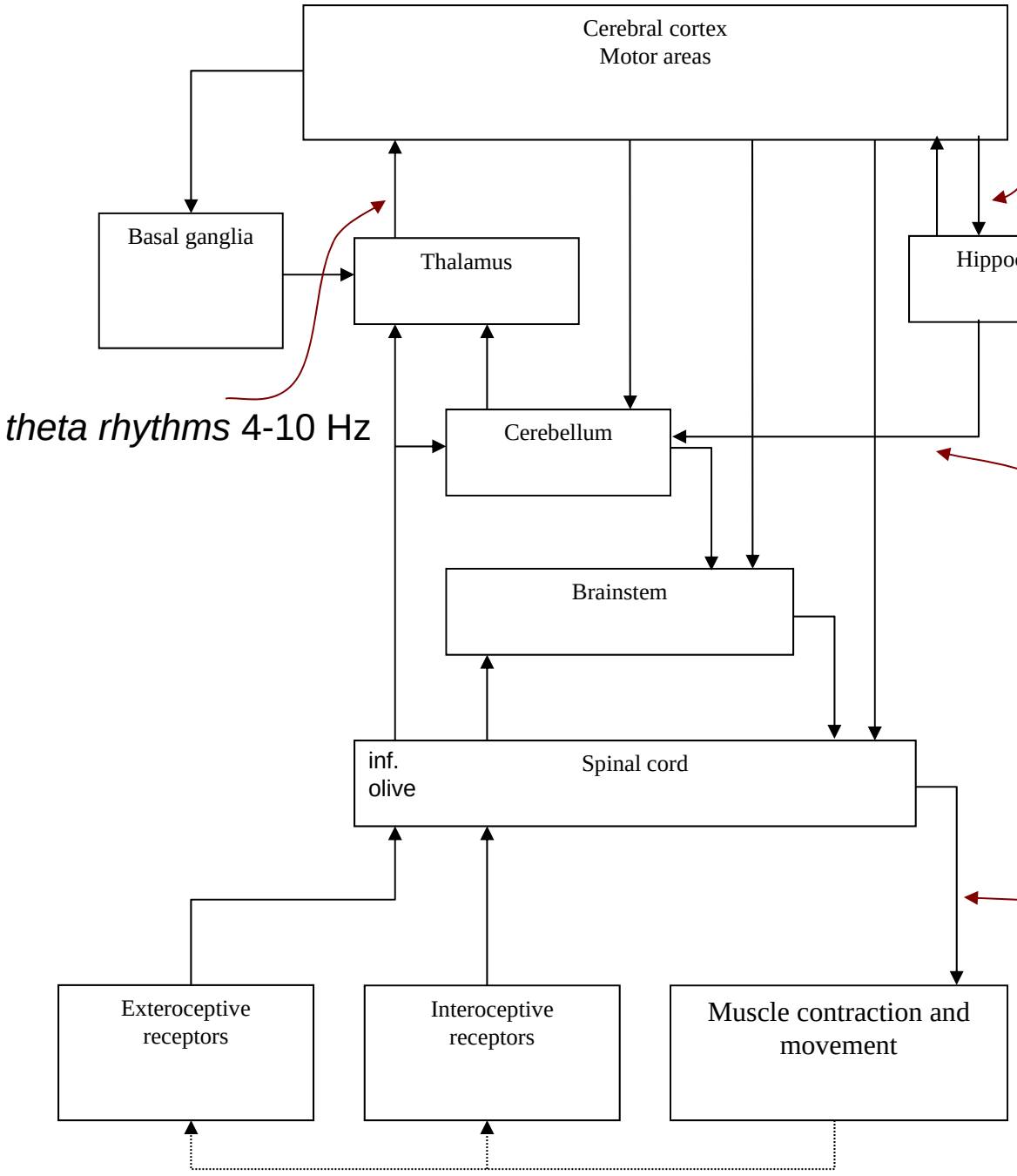
Adaptive Critic structure

Reinforcement learning

Theta waves 4-8 Hz



Motor control 200 Hz



gamma rhythms 30-100 Hz

Intense processing due to the amounts of information data to be processed

Cognitive map of the environment
- place cells -

theta rhythms 4-10 Hz

theta rhythms 4-10 Hz

Behavior reference
Information sent to the lower processing levels

Motor control 200 Hz

Kyriakos Vamvoudakis

2. Synchronous
Online Solution of Optimal Control for Nonlinear Systems

Optimal Adaptive Control

Policy Iteration gives the structure needed for online optimal solution

A new structure of adaptive controllers

CT Policy Iteration – a Reinforcement Learning Technique

To avoid solving HJB equation $0 = \left(\frac{dV^*}{dx}\right)^T f + Q(x) - \frac{1}{4}\left(\frac{dV^*}{dx}\right)^T g R^{-1} g^T \frac{dV^*}{dx}$

Utility $r(x,u) = Q(x) + u^T R u$

Cost for any given admissible *u(x)*

$$0 = \left(\frac{\partial V}{\partial x}\right)^T f(x,u) + r(x,u) \equiv H(x, \frac{\partial V}{\partial x}, u)$$

CT Bellman equation

Policy Iteration Solution

Pick stabilizing initial control policy

Policy Evaluation - Find cost, Bellman eq.

$$0 = \left(\frac{\partial V_j}{\partial x}\right)^T f(x, h_j(x)) + r(x, h_j(x))$$

$$V_j(0) = 0$$

Policy improvement - Update control

$$h_{j+1}(x) = -\tfrac{1}{2} R^{-1} g^T(x) \frac{\partial V_j}{\partial x}$$

- Convergence proved by Leake and Liu 1967, Saridis 1979 if Lyapunov eq. solved exactly
- Beard & Saridis used Galerkin Integrals to solve Lyapunov eq.
- Abu Khalaf & Lewis used NN to approx. V for nonlinear systems and proved convergence

Full system dynamics must be known
Off-line solution

Kyriakos Vamvoudakis

Synchronous
Online Solution of Optimal Control for Nonlinear Systems

Optimal Adaptive Control

Policy Iteration gives the structure needed for online optimal solution

Need to solve online:

Bellman eq. for Value

$$0 = \dot{V} + r(x, h(x)) = \left(\frac{\partial V}{\partial x}\right)^T \dot{x} + r(x, h(x)) = \left(\frac{\partial V}{\partial x}\right)^T f(x, h(x)) + Q(x) + h^T R h \equiv H(x, \frac{\partial V}{\partial x}, h(x))$$

Control update

$$h(x) = -\frac{1}{2} R^{-1} g^T(x) \nabla V$$

Solve by parameterizing value *V(x)*

Value Function Approximation – ***Paul Werbos***

converts Bellman PDE into algebraic equation

Critic NN

Take VFA as $V(x) = W_1^T \phi_1(x) + \varepsilon(x)$, $\nabla V(x) = \nabla \phi_1^T W_1$

Then Bellman eq $0 = \left(\frac{\partial V}{\partial x}\right)^T (f + gu) + Q(x) + u^T R u \equiv H(x, \frac{\partial V}{\partial x}, u)$

becomes

$$H(x, W_1, u) = W_1^T \nabla \phi_1 (f + gu) + Q(x) + u^T R u = \varepsilon_H$$

PDE

Algebraic eq.

W_1= LS solution to this eq for given *N*. Unknown.

Action NN for Control Approximation

$$u(x) = -\tfrac{1}{2} R^{-1} g^T(x) \nabla \phi_1{}^T \hat{W}_2,$$

Comes from $u(x) = -\tfrac{1}{2} R^{-1} g^T(x) \nabla V$

$\nabla V(x) = \nabla \phi_1^T W_1$

Online Synchronous Policy Iteration

Theorem (Kyriakos Vamvoudakis)- Online Learning of Nonlinear Optimal Control

Let $\sigma_1 \equiv \nabla\phi_1(f+gu)$ be PE. Tune critic NN weights as

$$\dot{\hat{W}}_1 = -a_1 \frac{\partial E_1}{\partial \hat{W}_1} = -a_1 \frac{\sigma_1}{(\sigma_1^T\sigma_1+1)^2}[\sigma_1^T\hat{W}_1 + Q(x) + u^T Ru]$$

Learning the Value

Tune actor NN weights as

$$\dot{\hat{W}}_2 = -\alpha_2\{(F_2\hat{W}_2 - F_1\bar{\sigma}_1^T\hat{W}_1) - \frac{1}{4}\bar{D}_1(x)\hat{W}_2 m^T(x)\hat{W}_1\}$$

Learning the control policy

where $\bar{D}_1(x) \equiv \nabla\phi_1(x)g(x)R^{-1}g^T(x)\nabla\phi_1^T(x)$, $m \equiv \frac{\sigma_1}{(\sigma_1^T\sigma_1+1)^2}$

Then there exists an N_0 such that, for the number of hidden layer units $N > N_0$

the closed-loop system state, the critic NN error $\tilde{W}_1 = W_1 - \hat{W}_1$

and the actor NN error $\tilde{W}_2 = W_1 - \hat{W}_2$ are UUB bounded.

Summary Nota Bene

Control policy

$$u(x)=-\frac{1}{2}R^{-1}g^{T}(x)\nabla\phi_1{}^{T}\hat{W}_2$$

Tune critic NN weights as

$$\dot{\hat{W}}_1=-a_1\frac{\partial E_1}{\partial \hat{W}_1}=-a_1\frac{\sigma_1}{(\sigma_1^{T}\sigma_1+1)^2}[\sigma_1{}^{T}\hat{W}_1+Q(x)+u^{T}Ru]$$

Tune actor NN weights as

Extra terms needed for stability

$$\dot{\hat{W}}_2=-\alpha_2\{(F_2\hat{W}_2-F_1\bar{\sigma}_1^{T}\hat{W}_1)-\frac{1}{4}\overline{D}_1(x)\hat{W}_2m^{T}(x)\hat{W}_1\}$$

Note, it does not work to simply set

$$u(x)=-\frac{1}{2}R^{-1}g^{T}(x)\nabla\phi_1{}^{T}\hat{W}_1$$

Must have TWO NNs

Lyapunov energy-based Proof:

$$L(t) = V(x) + \frac{1}{2} tr(\tilde{W}_1^T a_1^{-1} \tilde{W}_1) + \frac{1}{2} tr(\tilde{W}_2^T a_2^{-1} \tilde{W}_2).$$

V(x)= Unknown solution to HJB eq.

$$0 = \left(\frac{dV}{dx}\right)^T f + Q(x) - \tfrac{1}{4}\left(\frac{dV}{dx}\right)^T g R^{-1} g^T \frac{dV}{dx}$$

Guarantees stability

$$\tilde{W}_1 = W_1 - \hat{W}_1$$

$$\tilde{W}_2 = W_1 - \hat{W}_2$$

W_1= Unknown LS solution to Bellman equation for given N

$$H(x, W_1, u) = W_1^T \nabla \phi_1 (f + gu) + Q(x) + u^T R u = \varepsilon_H$$

ONLINE solution
Does not require solution of HJB or nonlinear Lyapunov eq.

Does require system dynamics to be known

Finds approximate local smooth solution to NONLINEAR HJB equation online

An optimal adaptive controller
- ‘indirect’ because it identifies parameters for VFA
- ‘direct’ because control is directly found from value function

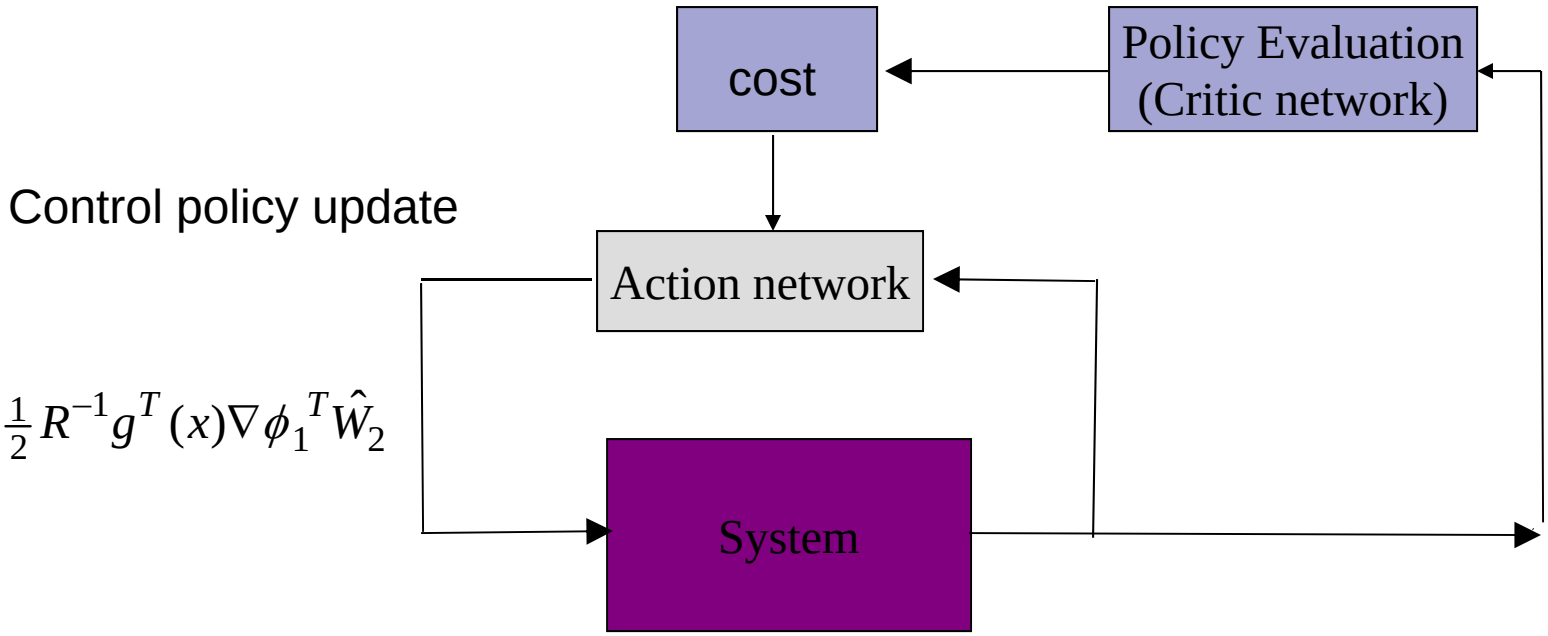
A New Adaptive Control Structure with Multiple Tuned Loops

Adaptive Critics

The Adaptive Critic Architecture

Value update- solve Bellman eq.

$$V(x) = W_1^T \phi_1(x)$$

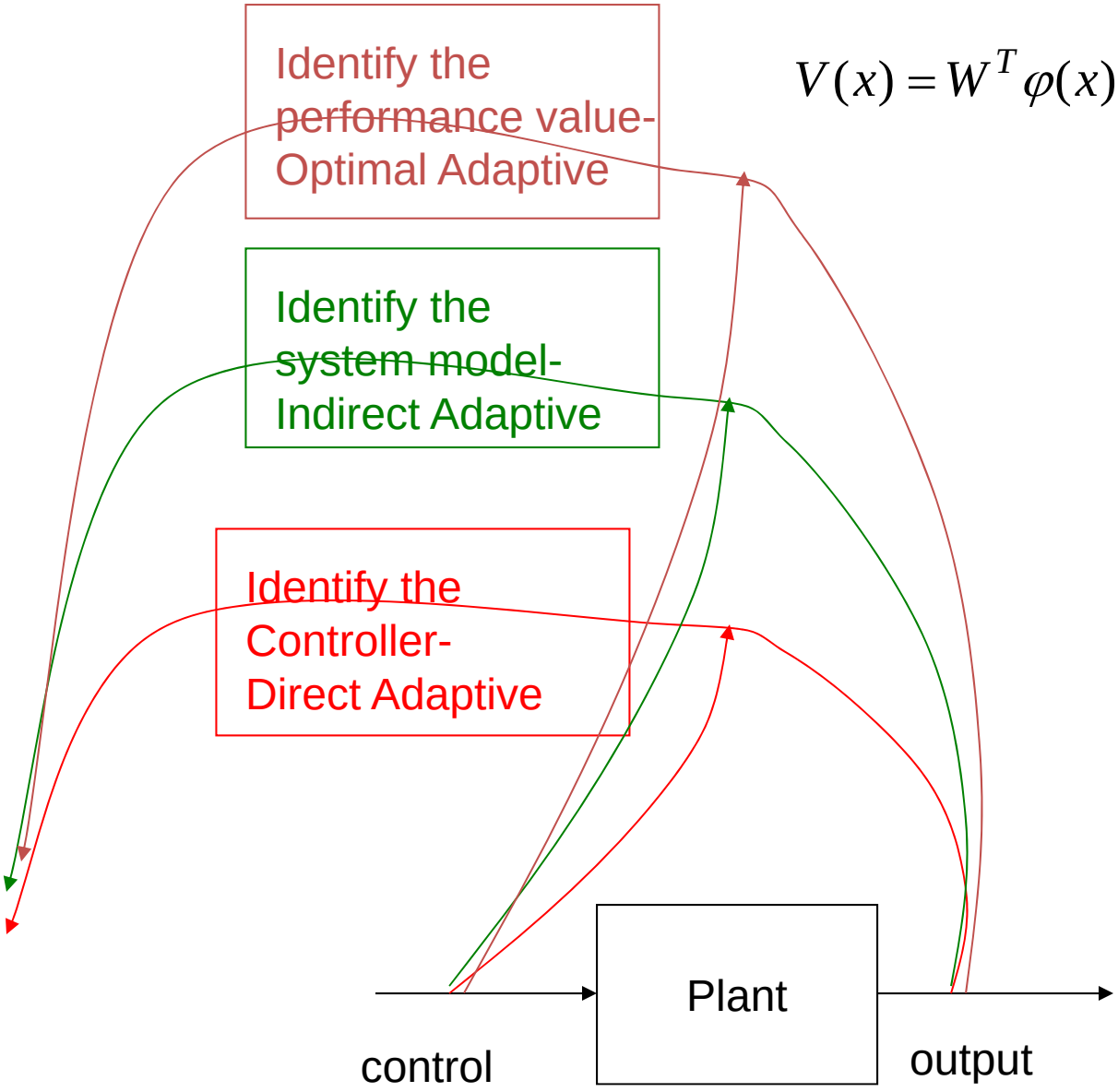


Critic and Actor tuned simultaneously

Leads to ONLINE FORWARD-IN-TIME implementation of optimal control

Optimal Adaptive Control

Adaptive Control



Simulation 1- F-16 aircraft pitch rate controller

$$\dot{x} = \begin{bmatrix} -1.01887 & 0.90506 & -0.00215 \\ 0.82225 & -1.07741 & -0.17555 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Stevens and Lewis 2003

$$x = [\alpha \quad q \quad \delta_e]$$

$$Q = I, \quad R = I$$

Solves ARE online

$$0 = PA + A^T P + Q - PBR^{-1}B^T P$$

Select quadratic NN basis set for VFA

Exact solution $W_1^* = [p_{11} \quad 2p_{12} \quad 2p_{13} \quad p_{22} \quad 2p_{23} \quad p_{33}]^T$

$= [1.4245 \quad 1.1682 \quad -0.1352 \quad 1.4349 \quad -0.1501 \quad 0.4329]^T$

Must add probing noise to get PE

$$u(x) = -\frac{1}{2} R^{-1} g^T(x) \nabla \phi_1^T \hat{W}_2 + n(t)$$

(exponentially decay *n(t)*)

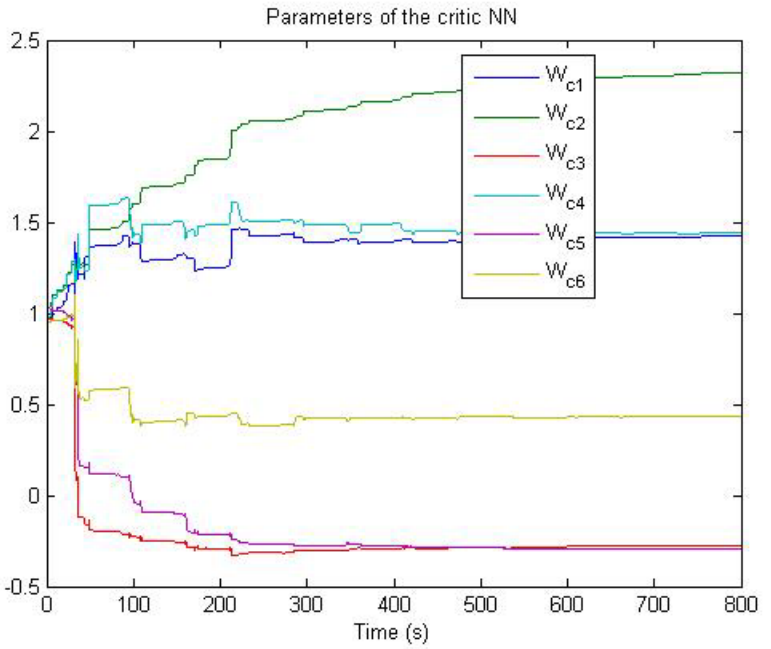
Algorithm converges to

$\hat{W}_1(t_f) = [1.4279 \quad 1.1612 \quad -0.1366 \quad 1.4462 \quad -0.1480 \quad 0.4317]^T$.

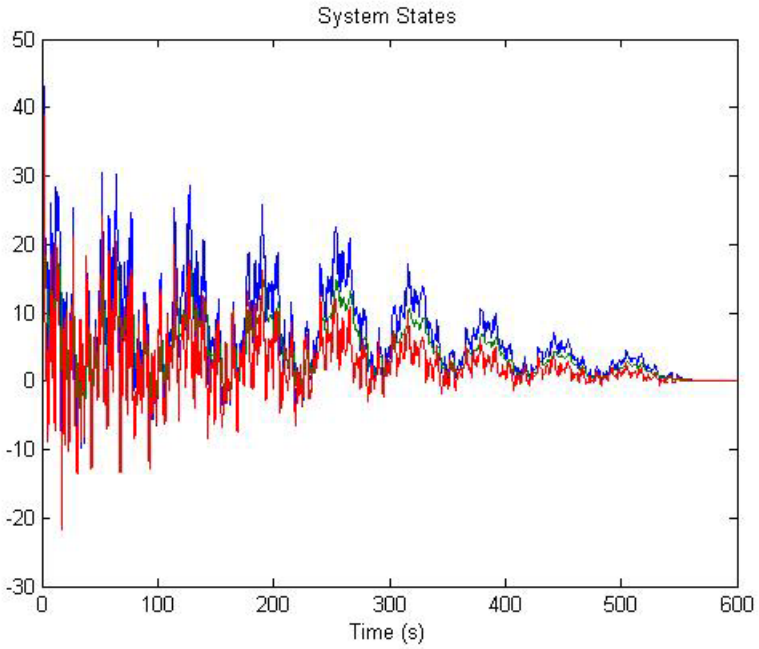
$\hat{W}_2(t_f) = [1.4279 \quad 1.1612 \quad -0.1366 \quad 1.4462 \quad -0.1480 \quad 0.4317]^T$

$$\hat{u}_2(x) = -\frac{1}{2} R^{-1} B^T P x = -\frac{1}{2} R^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 2x_1 & 0 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & 2x_2 & 0 \\ 0 & x_3 & x_2 \\ 0 & 0 & 2x_3 \end{bmatrix}^T \begin{bmatrix} 1.4279 \\ 1.1612 \\ -0.1366 \\ 1.4462 \\ -0.1480 \\ 0.4317 \end{bmatrix}$$

Critic NN parameters-
Converge to ARE solution



System states



Nevistic V. and Primbs J. A. (1996)
Converse optimal

Simulation 2. – Nonlinear System

$$\dot{x} = f(x) + g(x)u, \qquad x \in R^2$$

$$f(x) = \begin{bmatrix} -x_1 + x_2 \\ -0.5x_1 - 0.5x_2(1-(\cos(2x_1)+2)^2) \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ \cos(2x_1)+2 \end{bmatrix}.$$

$Q = I, \quad R = I$

Solves HJB equation online

$$0 = \left(\frac{dV^*}{dx}\right)^T f + Q(x) - \tfrac{1}{4}\left(\frac{dV^*}{dx}\right)^T g R^{-1} g^T \frac{dV^*}{dx}$$

Optimal Value $\quad V^*(x) = \frac{1}{2}x_1^2 + x_2^2$

Optimal control $\quad u^*(x) = -(\cos(2x_1)+2)x_2.$

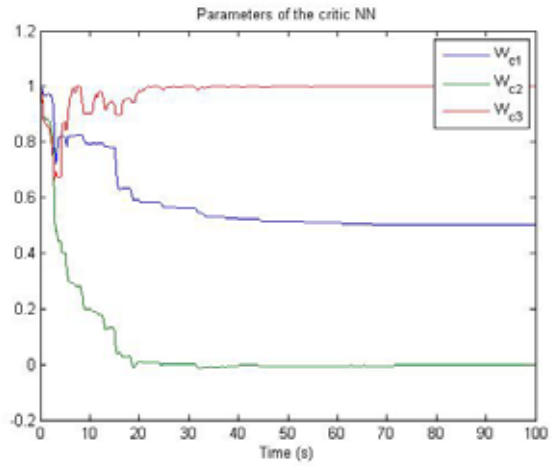
Select VFA basis set $\quad \phi_1(x) = [\; x_1^2 \quad x_1x_2 \quad x_2^2\;]^T,$

Algorithm converges to

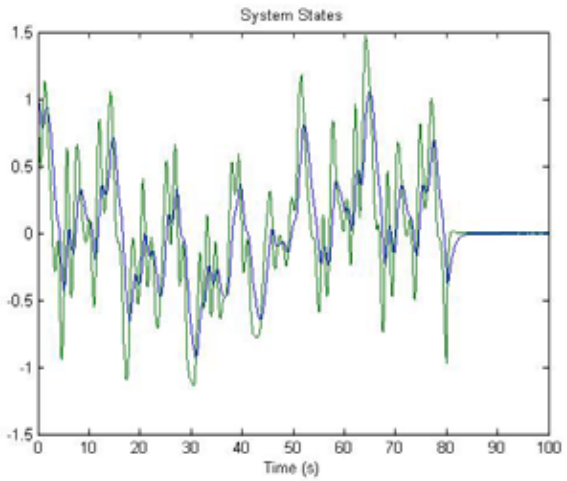
$\hat{W}_1(t_f) = [0.5017 \quad -0.0020 \quad 1.0008]^T.$

$\hat{W}_2(t_f) = [0.5017 \quad -0.0020 \quad 1.0008]^T.$

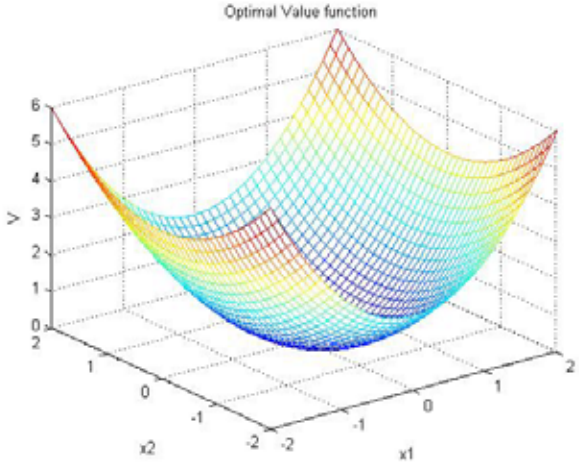
$$\hat{u}_2(x) = -\tfrac{1}{2}R^{-1}\begin{bmatrix} 0 \\ \cos(2x_1)+2 \end{bmatrix}^T \begin{bmatrix} 2x_1 & 0 \\ x_2 & x_1 \\ 0 & 2x_2 \end{bmatrix}^T \begin{bmatrix} 0.5017 \\ -0.0020 \\ 1.0008 \end{bmatrix}$$



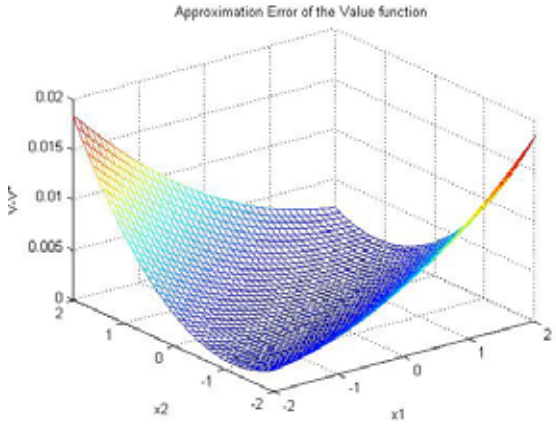
Critic NN parameters



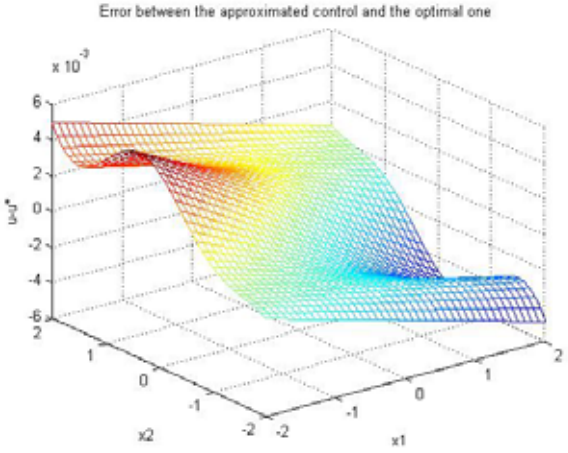
states



Optimal value fn.



Value fn. approx. error



Control approx error

Online Synchronous Policy Iteration using IRL

Does not need to know *f(x)*

Replace $\sigma_1 \equiv \nabla\phi_1(f+gu)$ by $\Delta\phi(x(t)) \equiv \phi(x(t)) - \phi(x(t-T))$

Theorem (Vamvoudakis & Vrabie)- Online Learning of Nonlinear Optimal Control

Let $\Delta\phi(x(t)) \equiv \phi(x(t)) - \phi(x(t-T))$ be PE. Tune critic NN weights as

$$\dot{\hat{W}}_1 = -a_1 \frac{\Delta\phi(x(t))^T}{\left(1+\Delta\phi(x(t))^T \Delta\phi(x(t))\right)^2}\left(\Delta\phi(x(t))^T \hat{W}_1 + \int_{t-T}^{t}\left(Q(x)+\frac{1}{4}\hat{W}_2{}^T \overline{D}_1 \hat{W}_2\right)d\tau\right)$$

Learning the Value

Tune actor NN weights as

$$\dot{\hat{W}}_2 = -a_2\left(F_2\hat{W}_2 - F_1\Delta\phi(x(t))^T\hat{W}_1\right) - \tfrac{1}{4}a_2\overline{D}_1(x)\hat{W}_2\frac{\Delta\phi(x(t))^T}{\left(1+\Delta\phi(x(t))^T\Delta\phi(x(t))\right)^2}\hat{W}_1$$

Learning the control policy

Then there exists an N_0 such that, for the number of hidden layer units $N > N_0$

the closed-loop system state, the critic NN error $\tilde{W}_1 = W_1 - \hat{W}_1$

and the actor NN error $\tilde{W}_2 = W_1 - \hat{W}_2$ are UUB bounded.

Can avoid knowledge of drift term *f(x) by* using Integral Reinforcement Learning (IRL)

Draguna Vrabie

Double Policy Iteration Algorithm to Solve HJI

Add inner loop to solve for available storage

Start with stabilizing initial policy $u_0(x)$

1. For a given control policy $u_j(x)$ solve for the value $V_{j+1}(x(t))$

2. Set $d^0 = 0$. For i=0,1,… solve for $V_j^i(x(t)), d^{i+1}$

$$0 = h^T h + \nabla V_j^{iT}(x)(f + gu_j + kd^i) + u_j{}^T R u_j - \gamma^2 \left\| d^i \right\|^2$$

ZS game Bellman eq.

$$d^{i+1} = \frac{1}{2\gamma^2} k^T(x) \nabla V_j^i$$

On convergence set $V_{j+1}(x) = V_j^i(x)$

3. Improve policy:

$$u_{j+1}(x) = -\frac{1}{2} R^{-1} g^T(x) \nabla V_{j+1}$$

- Convergence proved by Van der Schaft if can solve nonlinear Lyapunov equation exactly
- Abu Khalaf & Lewis used NN to approximate V for nonlinear systems and proved convergence

Off-line solution

Nonlinear Lyapunov equation must be solved at each step

Kyriakos Vamvoudakis

Online Solution of ZS Games for Nonlinear Systems

Optimal (Game) Adaptive Control

Policy Iteration gives the structure needed for online solution

Need to solve online these 3 equations:

ZS game Bellman eq. for Value

$$0 = h^T h + \nabla V^T(x)(f + gu + kd) + u^T R u - \gamma^2 \|d\|^2$$

Disturbance update

$$d = \frac{1}{2\gamma^2} k^T(x) \nabla V$$

Control update

$$u(x) = -\tfrac{1}{2} R^{-1} g^T(x) \nabla V$$

Use Three Neural Networks

Critic NN for VFA $\hat{V}(x) = \hat{W}_1^T \phi_1(x)$

Bellman eq becomes algebraic eq.

$$H(x, \hat{W}_1, u) = \hat{W}_1^T \nabla \phi_1 (f + gu + kd) + h^T h + u^T R u - \gamma^2 \left\| d \right\|^2 = e_1$$

Control Actor NN

$u(x) = -\frac{1}{2} R^{-1} g^T (x) \nabla \phi_1^T \hat{W}_2$ Comes from $u(x) = -\frac{1}{2} R^{-1} g^T (x) \nabla V$

Disturbance actor NN

$d(x) = \frac{1}{2\gamma^2} k^T (x) \nabla \phi_1^T \hat{W}_3,$ Comes from $d = \frac{1}{2\gamma^2} k^T (x) \nabla V$

Simultaneously:

a. Solve Bellman eq.

and

b. update *u(x), d(x)*

Online Synchronous Policy Iteration for ZS games

Theorem (Kyriakos Vamvoudakis)- Online Gaming

Let $\sigma_2 = \nabla\phi_1(f + gu + kd)$ be PE. Tune critic NN weights as

$$\dot{\hat{W}}_1 = -a_1 \frac{\sigma_2}{(\sigma_2^T\sigma_2 + 1)^2}[\sigma_2^T\hat{W}_1 + h^T Qh - \gamma^2\|d\|^2 + u^T Ru]$$

Learning the Value

Tune actor NN weights as

$$\dot{\hat{W}}_2 = -\alpha_2\{(F_2\hat{W}_2 - F_1\bar{\sigma}_2^T\hat{W}_1) - \frac{1}{4}\bar{D}_1(x)\hat{W}_2 m^T(x)\hat{W}_1\}$$

$$\dot{\hat{W}}_3 = -\alpha_3\left\{\left(F_4\hat{W}_3 - F_3\bar{\sigma}_2^T\hat{W}_1\right) + \frac{1}{4\gamma^2}\bar{E}_1(x)\hat{W}_3 m^T\hat{W}_1\right\}$$

Learning the control policies

where $\bar{D}_1(x) \equiv \nabla\phi_1(x)g(x)R^{-1}g^T(x)\nabla\phi_1^T(x)$, $\bar{E}_1(x) \equiv \nabla\phi_1(x)kk^T\nabla\phi_1^T(x)$, $m \equiv \frac{\sigma_2}{(\sigma_2^T\sigma_2 + 1)^2}$

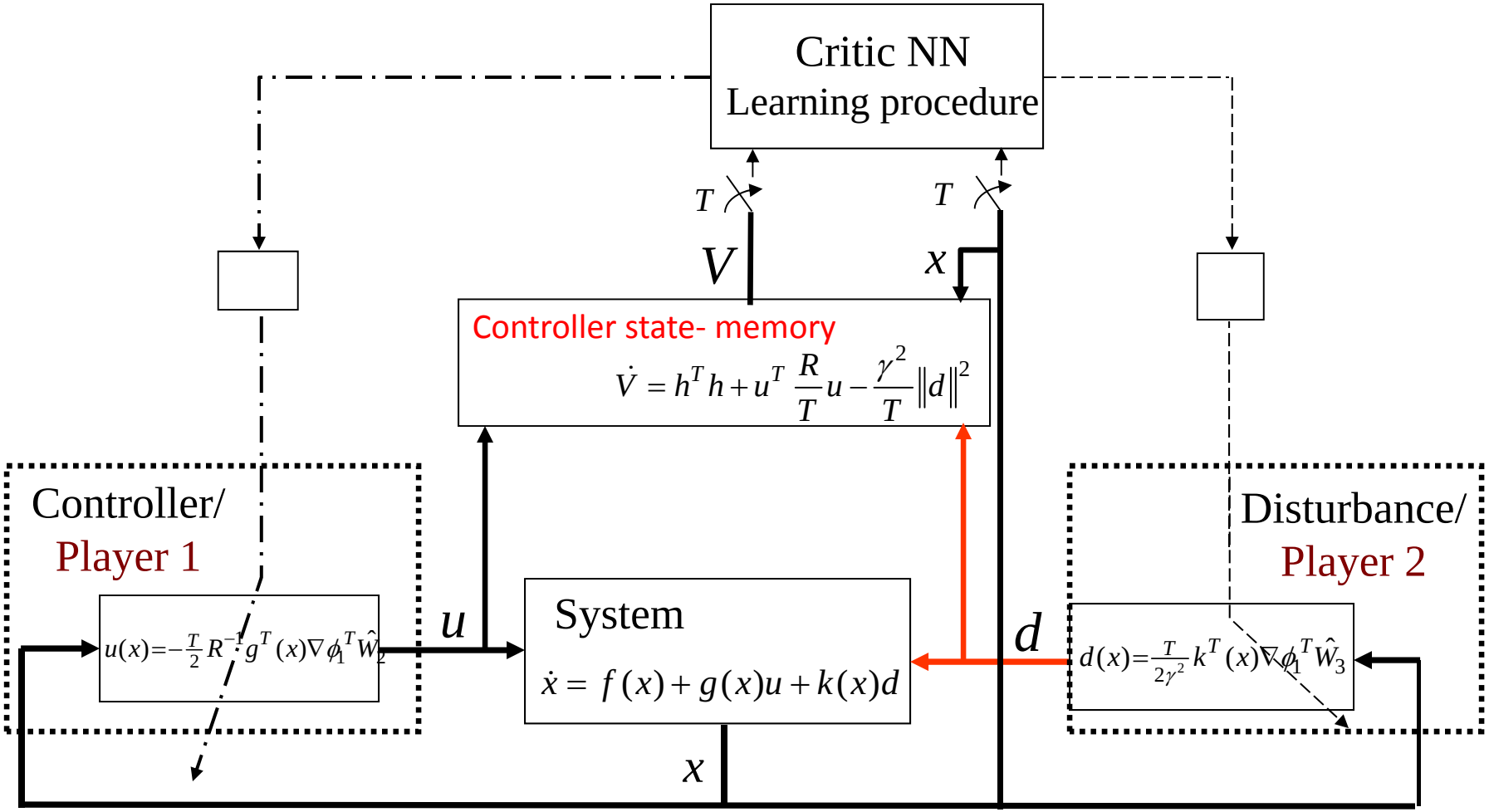
Then there exists an N_0 such that, for the number of hidden layer units $N > N_0$

the closed-loop system state, the critic NN error $\tilde{W}_1 = W_1 - \hat{W}_1$

and the actor NN errors $\tilde{W}_2 = W_1 - \hat{W}_2$, $\tilde{W}_3 = W_1 - \hat{W}_3$

are UUB bounded.

Actor-Critic structure –
A New Adaptive Controller with three tuned loops



A novel form of Hybrid Controller

ONLINE solution
Does not require solution of HJI eq, HJ eq, or nonlinear Lyapunov eq.

Does require system dynamics to be known

Finds approximate local smooth solution to NONLINEAR HJI equation online

An optimal adaptive controller
- ‘indirect’ because it identifies parameters for VFA
- ‘direct’ because control is directly found from value function

Simulation 1- F-16 aircraft pitch rate controller

$$\dot{x} = \begin{bmatrix} -1.01887 & 0.90506 & -0.00215 \\ 0.82225 & -1.07741 & -0.17555 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} d$$

Stevens and Lewis 2003

$x = [\alpha \quad q \quad \delta_e]$

$y = C^T x$

$Q = C^T C = I, \quad R = I$

Wind gust

Solves GARE online $A^T P + PA + Q - PBR^{-1}B^T P + \frac{1}{\gamma^2} PKK^T P = 0$

Exact solution $W_1^* = [p_{11} \quad 2p_{12} \quad 2p_{13} \quad p_{22} \quad 2p_{23} \quad p_{33}]^T$

$= [1.6573 \quad 1.3954 \quad -0.1661 \quad 1.6573 \quad -0.1804 \quad 0.4371]^T$

Must add probing noise to *u(x)* and *d(x)* to get PE

$$u(x) = -\frac{1}{2} R^{-1} g^T(x) \nabla \phi_1^T \hat{W}_2 + n(t)$$

(exponentially decay *n(t)*)

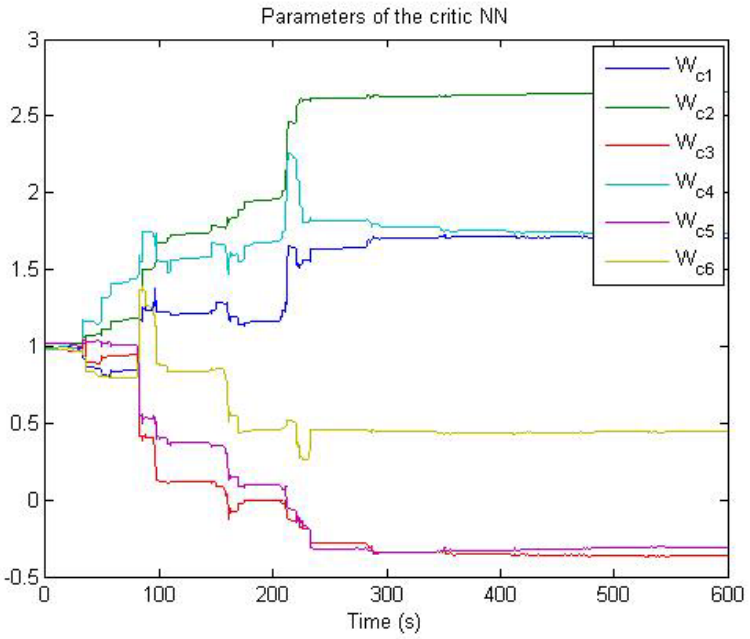
Algorithm converges to $\hat{W}_1(t_f) = [1.7090 \quad 1.3303 \quad -0.1629 \quad 1.7354 \quad -0.1730 \quad 0.4468]^T$.

$\hat{W}_2(t_f) = \hat{W}_3(t_f) = \hat{W}_1(t_f)$

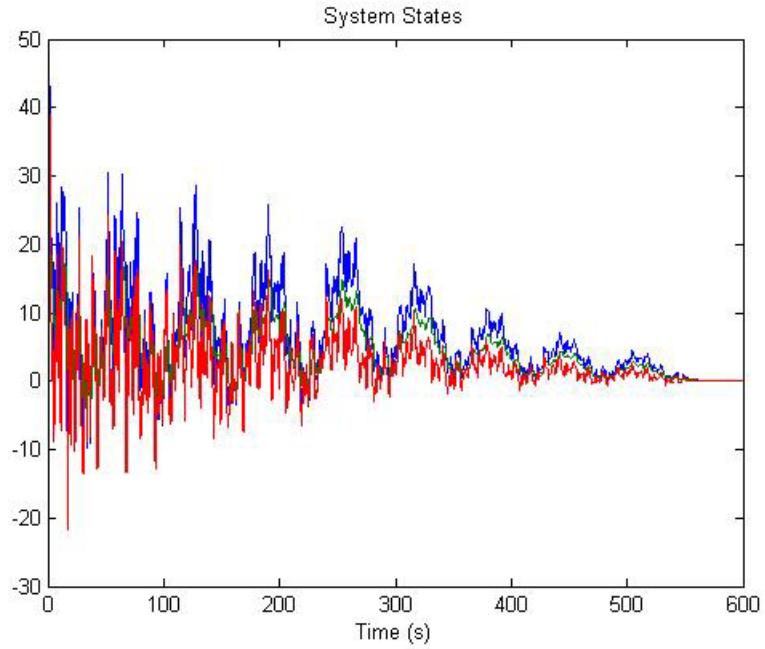
$$\hat{u}_2(x) = -\frac{1}{2} R^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 2x_1 & 0 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & 2x_2 & 0 \\ 0 & x_3 & x_2 \\ 0 & 0 & 2x_3 \end{bmatrix}^T \begin{bmatrix} 1.7090 \\ 1.3303 \\ -0.1629 \\ 1.7354 \\ -0.1730 \\ 0.4468 \end{bmatrix}$$

$$\hat{d}(x) = \frac{1}{2\gamma^2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 2x_1 & 0 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & 2x_2 & 0 \\ 0 & x_3 & x_2 \\ 0 & 0 & 2x_3 \end{bmatrix}^T \begin{bmatrix} 1.7090 \\ 1.3303 \\ -0.1629 \\ 1.7354 \\ -0.1730 \\ 0.4468 \end{bmatrix}$$

Critic NN parameters

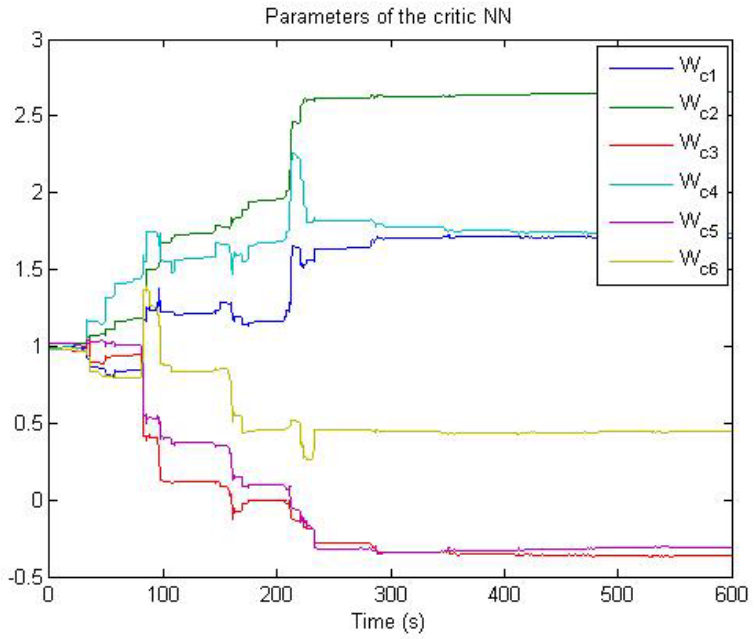


System states

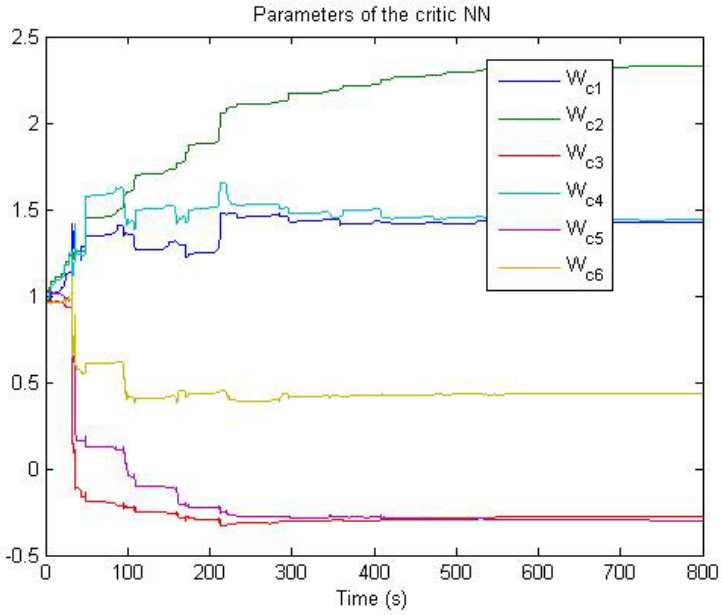


F-16 aircraft pitch rate controller

Critic NN parameters
With disturbance



Critic NN parameters
Without disturbance



Converges FASTER with an opponent
One learns faster with an adversary

Simulation 3. – Nonlinear System

$$\dot{x} = f(x) + g(x)u + k(x)d, \quad x \in R^2$$

$$f(x) = \begin{bmatrix} -x_1 + x_2 \\ -x_1^3 - x_2^3 + 0.25x_2(\cos(2x_1)+2)^2 - 0.25x_2\frac{1}{\gamma^2}(\sin(4x_1)+2)^2 \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ \cos(2x_1)+2 \end{bmatrix}, \quad k(x) = \begin{bmatrix} 0 \\ (\sin(4x_1)+2) \end{bmatrix}.$$

$$Q = I, \quad R = I, \quad \gamma = 8$$

Optimal Value $V^*(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$

Saddle point solution $u^*(x) = -(\cos(2x_1)+2)x_2.$ $d^*(x) = \frac{1}{\gamma^2}(\sin(4x_1)+2)x_2$

Solves HJI eq. online $0 = h^T h + \nabla V^T(x) f(x) - \frac{1}{4}\nabla V^T(x) g(x) R^{-1} g^T(x) \nabla V(x) + \frac{1}{4\gamma^2}\nabla V^T(x) k k^T \nabla V(x)$

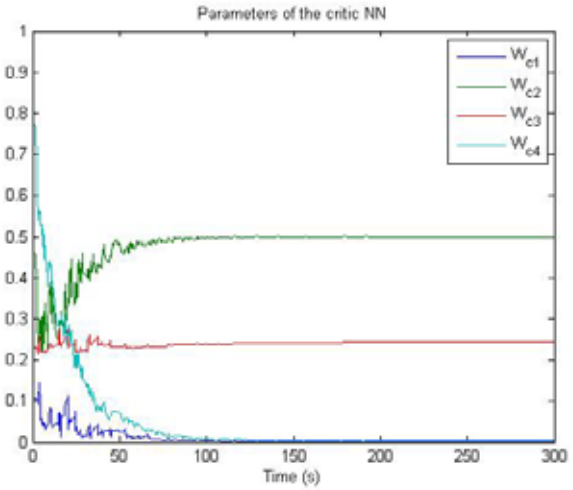
Select VFA basis set $\varphi_1(x) = [x_1^2 \quad x_2^2 \quad x_1^4 \quad x_2^4]$

Algorithm converges to $\hat{W}_1(t_f) = [0.0008 \quad 0.4999 \quad 0.2429 \quad 0.0032]^T$

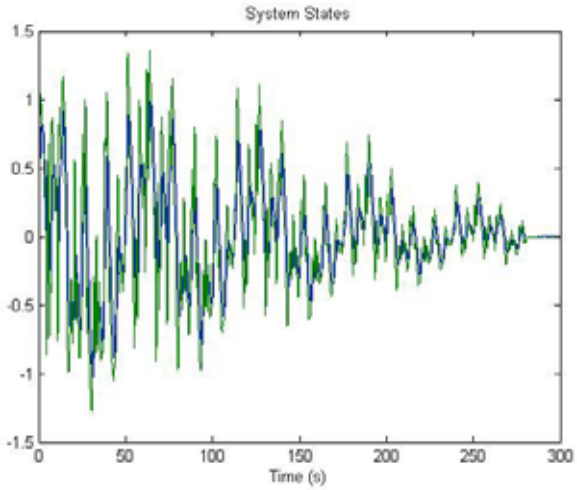
$\hat{W}_2(t_f) = \hat{W}_3(t_f) = \hat{W}_1(t_f)$

$$\hat{u}_2(x) = -\frac{1}{2}R^{-1}\begin{bmatrix} 0 \\ \cos(2x_1)+2 \end{bmatrix}^T \begin{bmatrix} 2x_1 & 0 \\ 0 & 2x_2 \\ 4x_1^3 & 0 \\ 0 & 4x_2^3 \end{bmatrix}^T \begin{bmatrix} 0.0008 \\ 0.4999 \\ 0.2429 \\ 0.0032 \end{bmatrix}$$

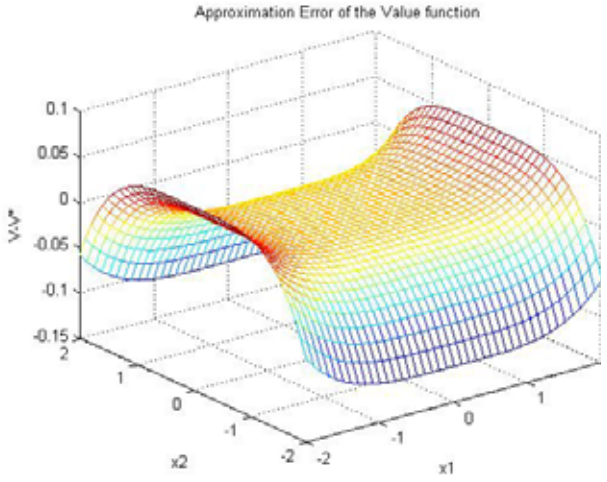
$$\hat{d}(x) = \frac{1}{2\gamma^2}\begin{bmatrix} 0 \\ \sin(4x_1)+2 \end{bmatrix}^T \begin{bmatrix} 2x_1 & 0 \\ 0 & 2x_2 \\ 4x_1^3 & 0 \\ 0 & 4x_2^3 \end{bmatrix}^T \begin{bmatrix} 0.0008 \\ 0.4999 \\ 0.2429 \\ 0.0032 \end{bmatrix}$$



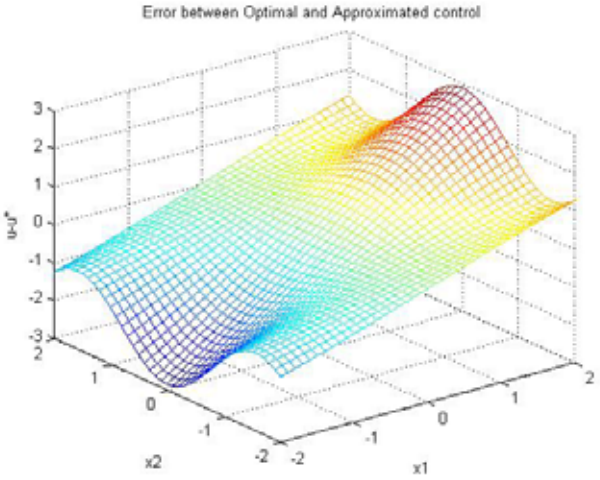
Critic NN parameters



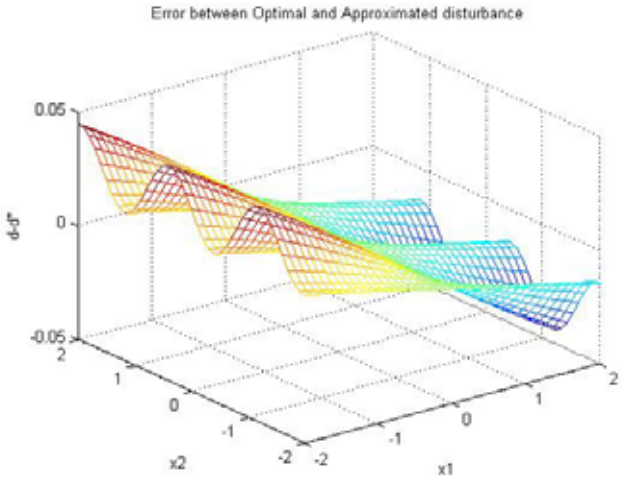
states



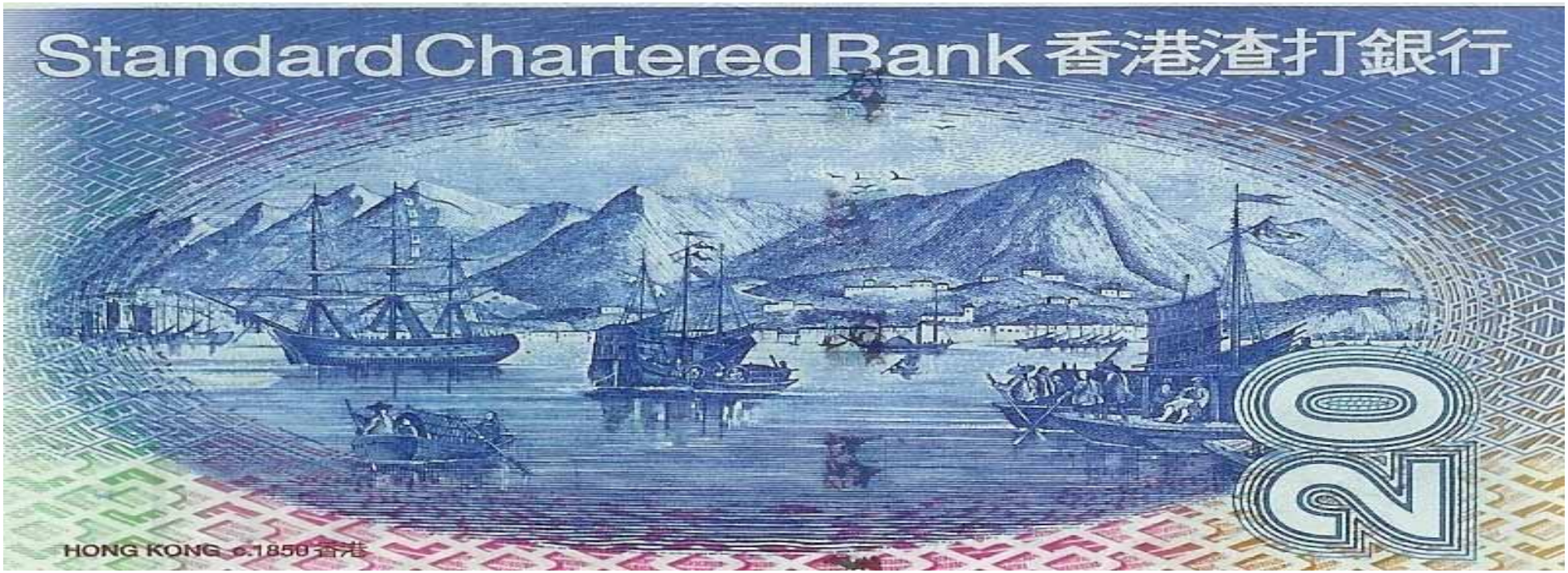
Value fn. approx. error

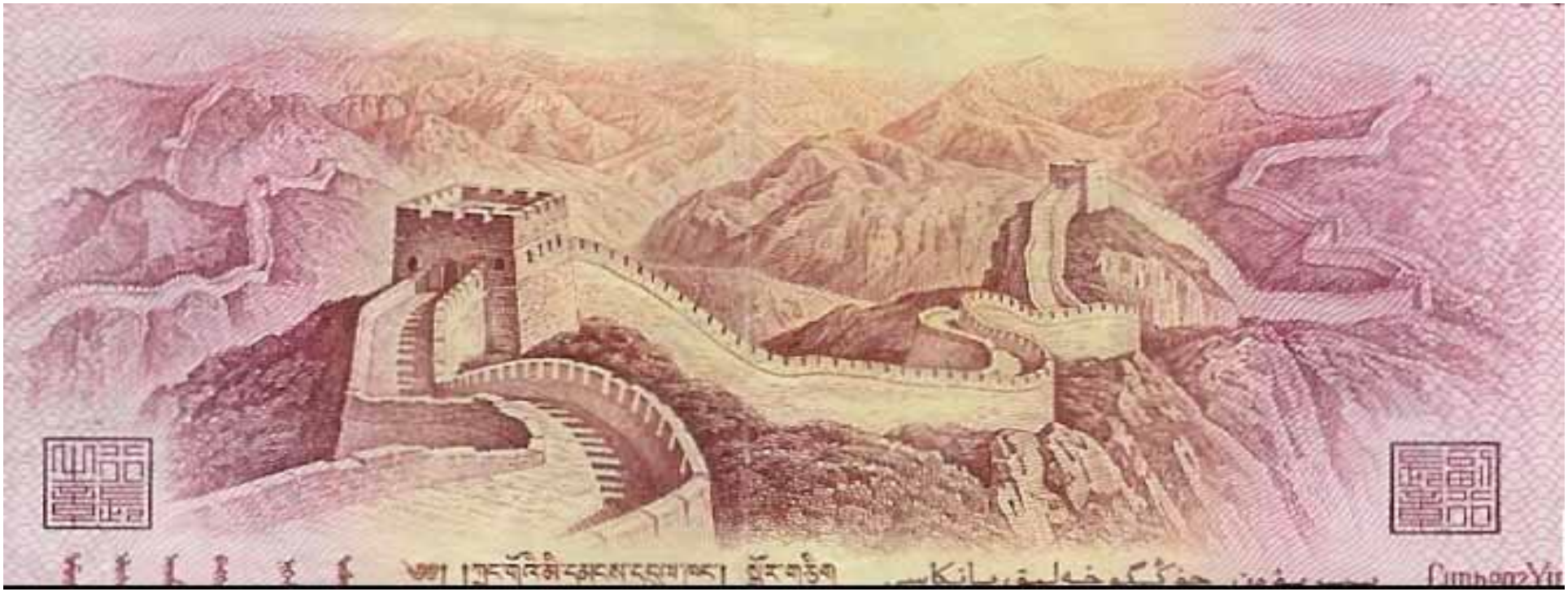


Control approx error



Dist. approx error


Standard Chartered Bank 香港渣打銀行
HONG KONG c.1850 香港
20



3. Real-Time Solution of Multi-Player NZS Games

Kyriakos Vamvoudakis

Multi-Player Nonlinear Systems $\dot{x} = f(x) + \sum_{j=1}^{N} g_j(x) u_j$ Continuous-time, *N* players

Optimal control $V_i^*(x(0), \mu_1, \mu_2, \ldots \mu_N) = \min_{\mu_i} \int_0^\infty (Q_i(x) + \sum_{j=1}^{N} \mu_i^T R_{ij} \mu_i)\, dt; \quad i \in N$

Nash equilibrium $V_i^* \triangleq V_i\ (\mu_1^*, \mu_2^*, \mu_i^*, ..., \mu_N^*) \le V_1\ (\mu_1^*, \mu_2^*, \mu_i, ..., \mu_N^*), i \in N$

Requires Offline solution of coupled Hamilton-Jacobi –Bellman eqs.

$$0 = (\nabla V_i)^T \left(f(x) - \tfrac{1}{2} \sum_{j=1}^{N} g_j(x) R_{jj}^{-1} g_j^T(x) \nabla V_j \right) + Q_i(x) + \tfrac{1}{4} \sum_{j=1}^{N} \nabla V_j{}^T g_j(x) R_{jj}^{-T} R_{ij} R_{jj}^{-1} g_j^T(x) \nabla V_j, \quad V_i(0) = 0$$

Control policies

$$\mu_i(x) = -\tfrac{1}{2} R_{ii}^{-1} g_i^T(x) \nabla V_i \quad , \ i \in N$$

Linear Quadratic Regulator Case- coupled AREs

$$0 = P_i A_c + A_c{}^T P_i + Q_i + \sum_{j=1}^{N} P_j B_j R_{jj}^{-T} R_{ij} R_{jj}^{-1} B_j^T P_j, \quad i \in N$$

These are hard to solve

In the nonlinear case, HJB generally cannot be solved

Team Interest vs. Self Interest

The objective functions of each player can be written as a *team average* term plus a *conflict of interest* term:

$$J_1 = \tfrac{1}{3}(J_1 + J_2 + J_3) + \tfrac{1}{3}(J_1 - J_2) + \tfrac{1}{3}(J_1 - J_3) \equiv J_{team} + J_1^{coi}$$

$$J_2 = \tfrac{1}{3}(J_1 + J_2 + J_3) + \tfrac{1}{3}(J_2 - J_1) + \tfrac{1}{3}(J_2 - J_3) \equiv J_{team} + J_2^{coi}$$

$$J_3 = \tfrac{1}{3}(J_1 + J_2 + J_3) + \tfrac{1}{3}(J_3 - J_1) + \tfrac{1}{3}(J_3 - J_2) \equiv J_{team} + J_3^{coi}$$

For N-players

$$J_i = \tfrac{1}{N}\sum_{j=1}^{N} J_j + \tfrac{1}{N}\sum_{j=1}^{N}(J_i - J_j) \equiv J_{team} + J_i^{coi}, \quad i = 1, N$$

For *N-player zero-sum games*, the first term is zero, i.e. the players have no goals in common.

Real-Time Solution of Multi-Player Games

Non-Zero Sum Games – Synchronous Policy Iteration

Kyriakos Vamvoudakis

Value functions $$V_i(x(0),\mu_1,\mu_2,\dots\mu_N)=\int_0^\infty (Q_i(x)+\sum_{j=1}^N \mu_i^T R_{ij}\mu_i)\,dt; \qquad i\in N$$

Leibniz gives Differential equivalent

Differential equivalent gives Bellman eqs.

$$0=Q_i(x)+\sum_{j=1}^N u_j^T R_{ij}u_j+(\nabla V_i)^T(f(x)+\sum_{j=1}^N g_j(x)u_j) \equiv H_i(x,\nabla V_i,u_1,\dots,u_N),\ i\in N$$

Policy Iteration Solution:

Solve Bellman eq. $$0=r(x,\mu^k{}_1,\dots,\mu^k{}_N)+(\nabla V_i^k)^T\left(f(x)+\sum_{j=1}^N g_j(x)\mu_j^i\right),\quad V^k{}_i(0)=0 \qquad i\in N$$

Policy Update $$\mu_i^{k+1}(x)=-\tfrac{1}{2}R_{ii}^{-1}g_i^T(x)\nabla V_i^k \quad ,\ i\in N$$

Convergence has not been proven
Hard to solve Hamiltonian equation
But this gives the structure we need for online Synchronous PI Solution

Policy Iteration gives the structure needed for online solution

Need to solve online:

Coupled Bellman eqs.

$$0 = Q_i(x) + \sum_{j=1}^{N} u_j^T R_{ij} u_j + (\nabla V_i)^T (f(x) + \sum_{j=1}^{N} g_j(x) u_j) \quad \equiv H_i(x, \nabla V_i, u_1, \ldots, u_N), \; i \in N$$

Control policies

$$\mu_i(x) = -\frac{1}{2} R_{ii}^{-1} g_i^T(x) \nabla V_i \quad , \; i \in N$$

Each player needs 2 NN – a Critic and an Actor

Real-Time Solution of Multi-Player Games

Kyriakos Vamvoudakis

Online Synchronous PI Solution for Multi-Player Games

Each player needs 2 NN – a Critic and an Actor

2-player case

N Critic Neural Networks for VFA

N Actor Neural Networks

Player 1

$$\hat{V}_1(x) = \hat{W}_1^T \phi_1(x),$$

$$u_1(x) = -\tfrac{1}{2} R_{11}^{-1} g_1^T(x) \nabla \phi_1^T \hat{W}_3,$$

Player 2

$$\hat{V}_2(x) = \hat{W}_2^T \phi_2(x)$$

$$u_2(x) = -\tfrac{1}{2} R_{22}^{-1} g_2^T(x) \nabla \phi_2^T \hat{W}_4$$

On-Line Learning – for Player 1:

$$\dot{\hat{W}}_1 = -a_1 \frac{\sigma_1}{(\sigma_1^T \sigma_1 + 1)^2} [\sigma_1^T \hat{W}_1 + Q_1(x) + u_1^T R_{11} u_1 + u_2^T R_{12} u_2]$$

Learns Bellman eq. solution

$$\dot{\hat{W}}_3 = -\alpha_3 \{(F_2 \hat{W}_3 - F_1 \bar{\sigma}_3^T \hat{W}_1) - \frac{1}{4} \nabla \phi_1 g(x) R_{11}^{-T} R_{21} R_{11}^{-1} g^T(x) \nabla \phi_1^T \hat{W}_3 m_2^T \hat{W}_2 - \frac{1}{4} \bar{D}_1(x) \hat{W}_3 m_1^T \hat{W}_1\}$$

Learns control policy

Lyapunov energy-based Proof:

$$L(t) = V(x) + \frac{1}{2} tr(\tilde{W}_1^T a_1^{-1} \tilde{W}_1) + \frac{1}{2} tr(\tilde{W}_2^T a_2^{-1} \tilde{W}_2).$$

V(x)= Unknown solution to HJB eq.

$$0 = \left(\frac{dV}{dx}\right)^T f + Q(x) - \tfrac{1}{4}\left(\frac{dV}{dx}\right)^T g R^{-1} g^T \frac{dV}{dx}$$

Guarantees stability

$$\tilde{W}_1 = W_1 - \hat{W}_1$$

$$\tilde{W}_2 = W_1 - \hat{W}_2$$

W_1= Unknown LS solution to Bellman equation for given N

$$H(x, W_1, u) = W_1^T \nabla \phi_1 (f + gu) + Q(x) + u^T R u = \varepsilon_H$$

Simulation. – Nonlinear System – 2-player game

$$\dot{x} = f(x) + g(x)u + k(x)d, \qquad x \in \mathbb{R}^2$$

$$f(x) = \begin{bmatrix} x_2 \\ -x_2 - \frac{1}{2}x_1 + \frac{1}{4}x_2(\cos(2x_1)+2)^2 + \frac{1}{4}x_2(\sin(4x_1^2)+2)^2 \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ \cos(2x_1)+2 \end{bmatrix}, \quad k(x) = \begin{bmatrix} 0 \\ (\sin(4x_1)+2) \end{bmatrix}.$$

$$Q_1 = 2Q_2 = 2I, \quad R_{11} = 2R_{22} = 2I, \quad R_{12} = 2R_{21} = 2I$$

Optimal Value s $\qquad V_1^*(x) = \frac{1}{2}x_1^2 + x_2^2 \qquad V_2^*(x) = \frac{1}{4}x_1^2 + \frac{1}{2}x_2^2$

Optimal Policies $\qquad u^*(x) = -2(\cos(2x_1)+2)x_2 \qquad d^*(x) = -(\sin(4x_1^2)+2)x_2$

Solves HJB equations online

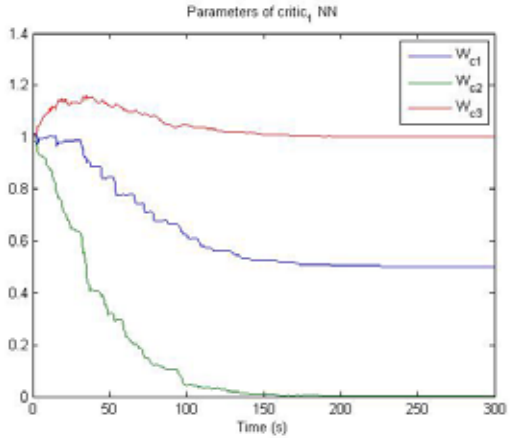
$$0 = (\nabla V_i)^T \left(f(x) - \frac{1}{2}\sum_{j=1}^{N} g_j(x) R_{jj}^{-1} g_j^T(x) \nabla V_j \right) + Q_i(x) + \frac{1}{4}\sum_{j=1}^{N} \nabla V_j{}^T g_j(x) R_{jj}^{-T} R_{ij} R_{jj}^{-1} g_j^T(x) \nabla V_j, \quad V_i(0) = 0$$

Select VFA basis set $\qquad \varphi_1(x) = \varphi_2(x) \equiv [x_1^2 \;\; x_1x_2 \;\; x_2^2]$

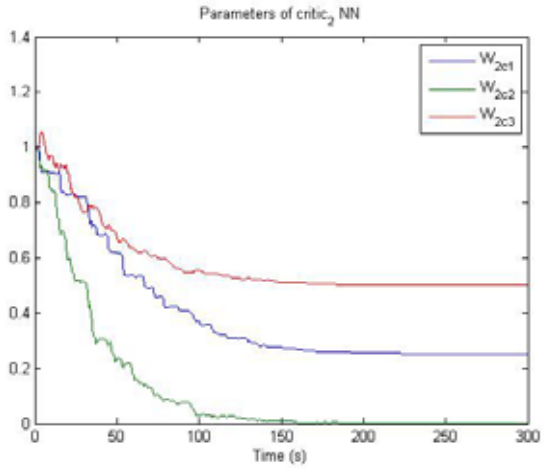
Algorithm converges to $\qquad \hat{W}_1(t_f) = [0.5015 \quad 0.0007 \quad 1.0001]^T = \hat{W}_3(t_f)$

$$\hat{W}_2(t_f) = [0.2514 \quad 0.0006 \quad 0.5001]^T = \hat{W}_4(t_f)$$

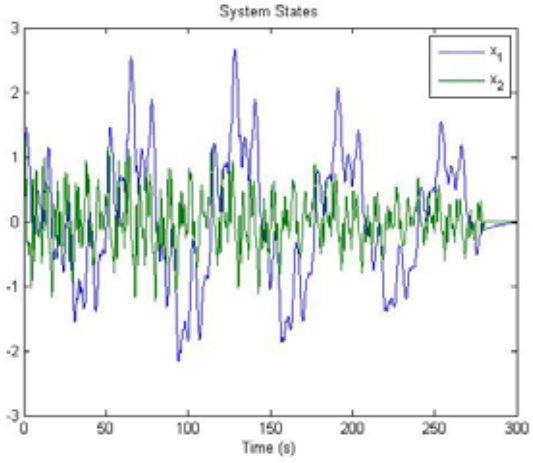
$$\hat{u}(x) = -\frac{1}{2}R_{11}^{-1}\begin{bmatrix} 0 \\ \cos(2x_1)+2 \end{bmatrix}^T \begin{bmatrix} 2x_1 & 0 \\ x_2 & x_1 \\ 0 & 2x_2 \end{bmatrix}^T \begin{bmatrix} 0.5015 \\ 0.0007 \\ 1.0001 \end{bmatrix} \qquad \hat{d}(x) = -\frac{1}{2}R_{22}^{-1}\begin{bmatrix} 0 \\ \sin(4x_1^2)+2 \end{bmatrix}^T \begin{bmatrix} 2x_1 & 0 \\ x_2 & x_1 \\ 0 & 2x_2 \end{bmatrix}^T \begin{bmatrix} 0.2514 \\ 0.0006 \\ 0.5001 \end{bmatrix}$$



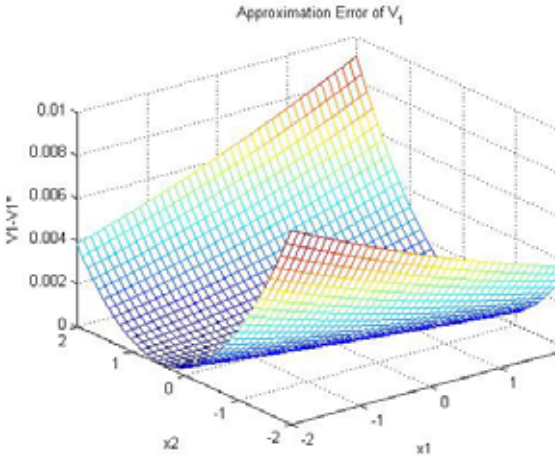
Critic 1 NN parameters



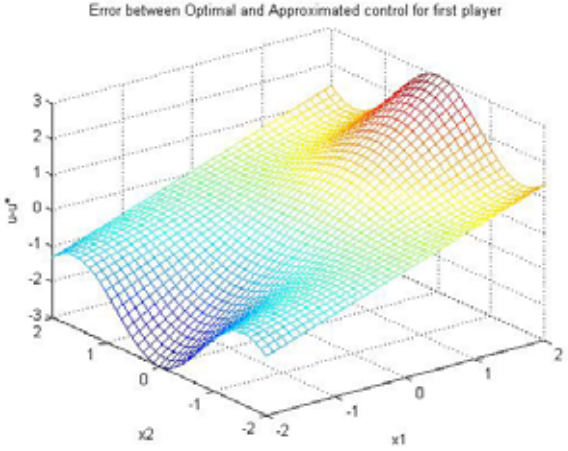
Critic 2 NN parameters



Evolution of the States



3D approximation error value for player 1.



3D approximation error of control for player 1.

Online Synchronous Policy Iteration using IRL

Does not need to know *f(x)*

Replace $\sigma_1 \equiv \nabla\phi_1(f+gu)$ by $\Delta\phi(x(t)) \equiv \phi(x(t)) - \phi(x(t-T))$

Theorem (Vamvoudakis & Vrabie)- Online Learning of Nonlinear Optimal Control

Let $\Delta\phi(x(t)) \equiv \phi(x(t)) - \phi(x(t-T))$ be PE. Tune critic NN weights as

$$\dot{\hat{W}}_1 = -a_1 \frac{\Delta\phi(x(t))^T}{\left(1+\Delta\phi(x(t))^T \Delta\phi(x(t))\right)^2}\left(\Delta\phi(x(t))^T \hat{W}_1 + \int_{t-T}^{t}\left(Q(x)+\frac{1}{4}\hat{W}_2{}^T \overline{D}_1 \hat{W}_2\right)d\tau\right)$$

Learning the Value

Tune actor NN weights as

$$\dot{\hat{W}}_2 = -a_2\left(F_2\hat{W}_2 - F_1\Delta\phi(x(t))^T\hat{W}_1\right) - \tfrac{1}{4}a_2\overline{D}_1(x)\hat{W}_2\frac{\Delta\phi(x(t))^T}{\left(1+\Delta\phi(x(t))^T\Delta\phi(x(t))\right)^2}\hat{W}_1$$

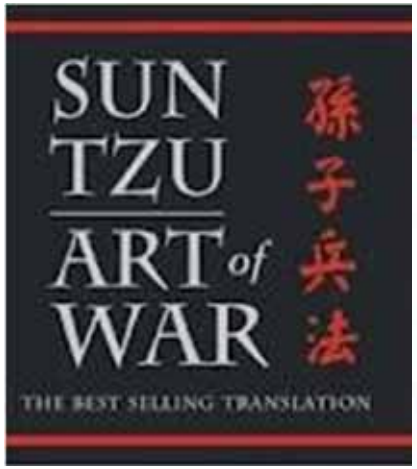
Learning the control policy

Then there exists an N_0 such that, for the number of hidden layer units $N > N_0$

the closed-loop system state, the critic NN error $\tilde{W}_1 = W_1 - \hat{W}_1$

and the actor NN error $\tilde{W}_2 = W_1 - \hat{W}_2$ are UUB bounded.

Graphical Coalitional Games

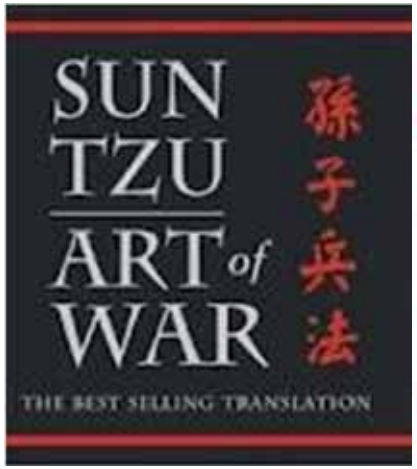


500 BC

孙子兵法

Sun Tz bin fa

Games on Communication Graphs



500 BC

孙子兵法

Sun Tz bin fa

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UTA Research Institute (UTARI)
The University of Texas at Arlington

Supported by :
NSF - PAUL WERBOS
ARO, AFOSR

Games on Communication Graphs

The University of Texas ARLINGTON

http://ARRI.uta.edu/acs

Books Coming

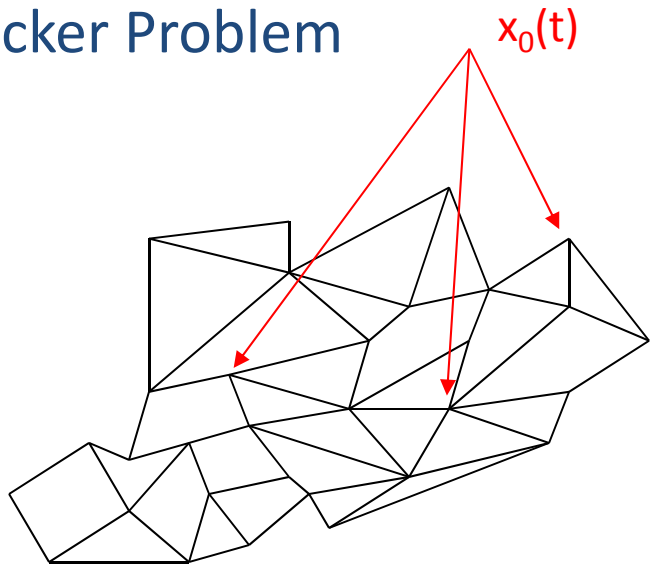
F.L. Lewis, H. Zhang, A. Das, K. Hengster-Movric, *Cooperative Control of Multi-Agent Systems: Optimal Design and Adaptive Control*, Springer-Verlag, 2013, to appear.

Key Point

Lyapunov Functions and Performance Indices
Must depend on graph topology

Hongwei Zhang, F.L. Lewis, and Abhijit Das
"Optimal design for synchronization of cooperative systems: state feedback, observer and output feedback,"
IEEE Trans. Automatic Control, vol. 56, no. 8, pp. 1948-1952, August 2011.

Graphical Games
Synchronization- Cooperative Tracker Problem



Node dynamics

$$\dot{x}_i = Ax_i + B_i u_i, \quad x_i(t) \in \mathbb{R}^n, \quad u_i(t) \in \mathbb{R}^{m_i}$$

Target generator dynamics

$$\dot{x}_0 = Ax_0$$

Synchronization problem

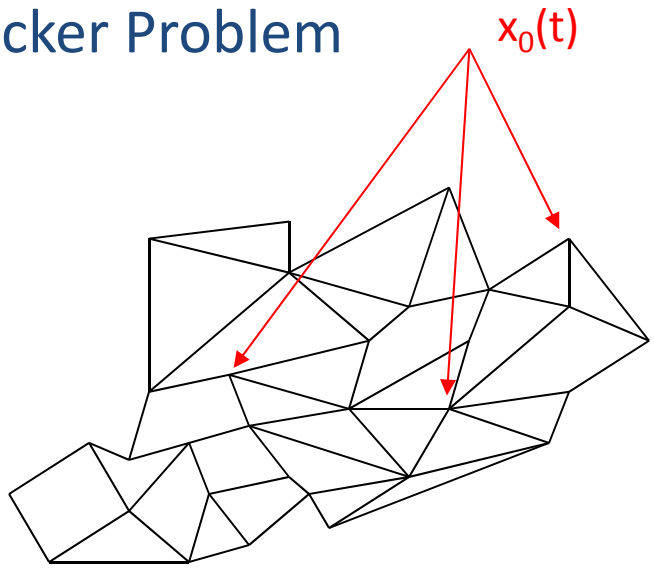
$$x_i(t) \to x_0(t), \forall i$$

Local neighborhood tracking error (Lihua Xie)

$$\delta_i = \sum_{j \in N_i} e_{ij}(x_i - x_j) + g_i(x_i - x_0),$$ Pinning gains $g_i \geq 0$ (Ron Chen)

K. Vamvoudakis and F.L. Lewis, "Graphical Games for Synchronization" Automatica, to appear.

Graphical Games
Synchronization- Cooperative Tracker Problem



Node dynamics

$$\dot{x}_i = Ax_i + B_i u_i, \quad x_i(t) \in \mathbb{R}^n, \quad u_i(t) \in \mathbb{R}^{m_i}$$

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$$\dot{x}_0 = Ax_0$$

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$$x_i(t) \to x_0(t), \forall i$$

Local neighborhood tracking error (Lihua Xie)

$$\delta_i = \sum_{j \in N_i} e_{ij}(x_i - x_j) + g_i(x_i - x_0), \quad \text{Pinning gains } g_i \geq 0 \quad \text{(Ron Chen)}$$

Standard way =

Global neighborhood tracking error

$$\delta = \begin{bmatrix} \delta_1^T & \delta_2^T & \cdots & \delta_N^T \end{bmatrix}^T \in \mathbb{R}^{nN}, \quad x = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_N^T \end{bmatrix}^T \in \mathbb{R}^{nN}, \quad \underline{x}_0 = \underline{I} x_0 \in \mathbb{R}^{nN}$$

$$\delta = \left((L+G) \otimes I_n\right)(x - \underline{x}_0) = \left((L+G) \otimes I_n\right)\zeta, \quad \zeta = (x - \underline{x}_0) \in \mathbb{R}^{nN}$$

Lemma. Let graph be strongly connected and at least one pinning gain nonzero. Then

$$\|\zeta\| \leq \|\delta\| / \underline{\sigma}(L+G)$$

and agents synchronize iff $\delta(t) \to 0$

Graphical Game: Games on Graphs

Kyriakos Vamvoudakis

Local nbhd. tracking error dynamics

$$\dot{\delta}_i = \sum_{j\in N_i} e_{ij}(\dot{x}_i - \dot{x}_j) \quad + g_i(\dot{x}_i - \dot{x}_0)$$

$$\dot{\delta}_i = A\delta_i + (d_i + g_i)B_i u_i - \sum_{j\in N_i} e_{ij}B_j u_j$$

Local agent dynamics driven by neighbors' controls

Define Local nbhd. performance index

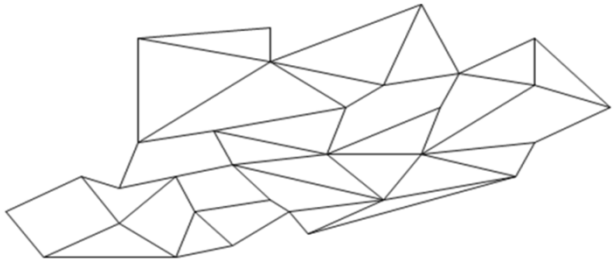
$$J_i(\delta_i(0), u_i, u_{-i}) = \tfrac{1}{2}\int_0^\infty (\delta_i^T Q_{ii}\delta_i + u_i^T R_{ii}u_i + \sum_{j\in N_i} u_j^T R_{ij}u_j)\,dt \quad \equiv \tfrac{1}{2}\int_0^\infty L_i(\delta_i(t), u_i(t), u_{-i}(t))\,dt$$

$$u_{-i}(t) \equiv \{u_j : j \in N_i\}$$

Values driven by neighbors' controls

Local value functions for fixed policies u_i

$$V_i(\delta_i(t)) = \tfrac{1}{2}\int_t^\infty (\delta_i^T Q_{ii}\delta_i + u_i^T R_{ii}u_i + \sum_{j\in N_i} u_j^T R_{ij}u_j)\,dt$$



Static Graphical Game

$$(G, U, v)$$

$$G = (V, E), \qquad v = \begin{bmatrix} v_1 & \cdots & v_N \end{bmatrix}^T$$

$$v_i(U_i, \{U_j : j \in N_i\}) \in R$$

Value depends only on neighbors

Standard N-player differential game

$$\dot{z} = Az + \sum_{i=1}^N B_i u_i$$

$$J_i(z(0), u_i, u_{-i}) = \tfrac{1}{2}\int_0^\infty (z^T Qz + \sum_{j=1}^N u_j^T R_{ij}u_j)\,dt$$

Dynamics depend on all other agents

Values depend on all other agents

Graphical Game: Games on Graphs

Kyriakos Vamvoudakis

Local nbhd. tracking error dynamics

$$\dot{\delta}_i = \sum_{j\in N_i} e_{ij}(\dot{x}_i - \dot{x}_j) \quad + g_i(\dot{x}_i - \dot{x}_0)$$

$$\dot{\delta}_i = A\delta_i + (d_i + g_i)B_i u_i - \sum_{j\in N_i} e_{ij}B_j u_j$$

Local agent dynamics driven by neighbors' controls

Define Local nbhd. performance index

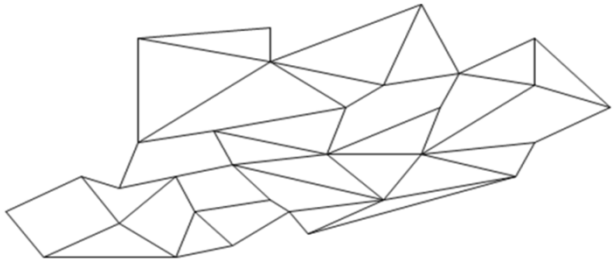
$$J_i(\delta_i(0), u_i, u_{-i}) = \tfrac{1}{2}\int_0^\infty (\delta_i^T Q_{ii}\delta_i + u_i^T R_{ii}u_i + \sum_{j\in N_i} u_j^T R_{ij}u_j)\,dt \quad \equiv \tfrac{1}{2}\int_0^\infty L_i(\delta_i(t), u_i(t), u_{-i}(t))\,dt$$

$$u_{-i}(t) \equiv \{u_j : j \in N_i\}$$

Values driven by neighbors' controls

Local value functions for fixed policies u_i

$$V_i(\delta_i(t)) = \tfrac{1}{2}\int_t^\infty (\delta_i^T Q_{ii}\delta_i + u_i^T R_{ii}u_i + \sum_{j\in N_i} u_j^T R_{ij}u_j)\,dt$$



Static Graphical Game

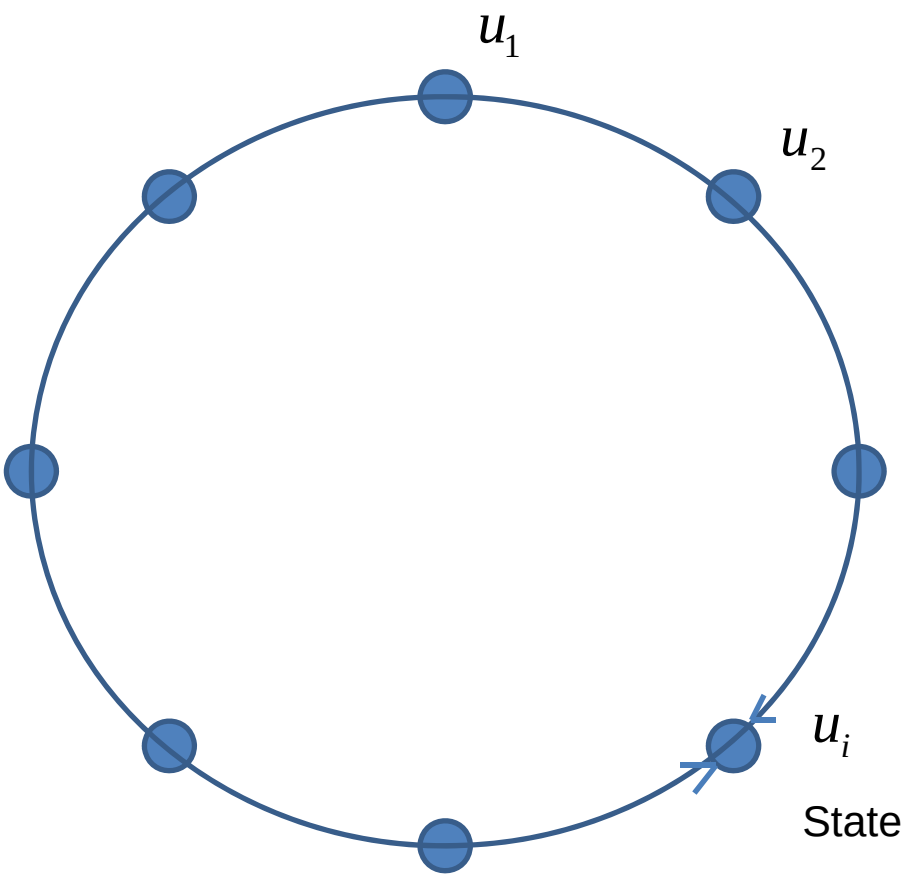
$$(G, U, v)$$

$$G = (V, E), \qquad v = \begin{bmatrix} v_1 & \cdots & v_N \end{bmatrix}^T$$

$$v_i(U_i, \{U_j : j \in N_i\}) \in R$$

Value depends only on neighbors

New Differential Graphical Game



Local Dynamics
Local Value Function
Only depends on
graph neighbors

u_i Control action of player *i*

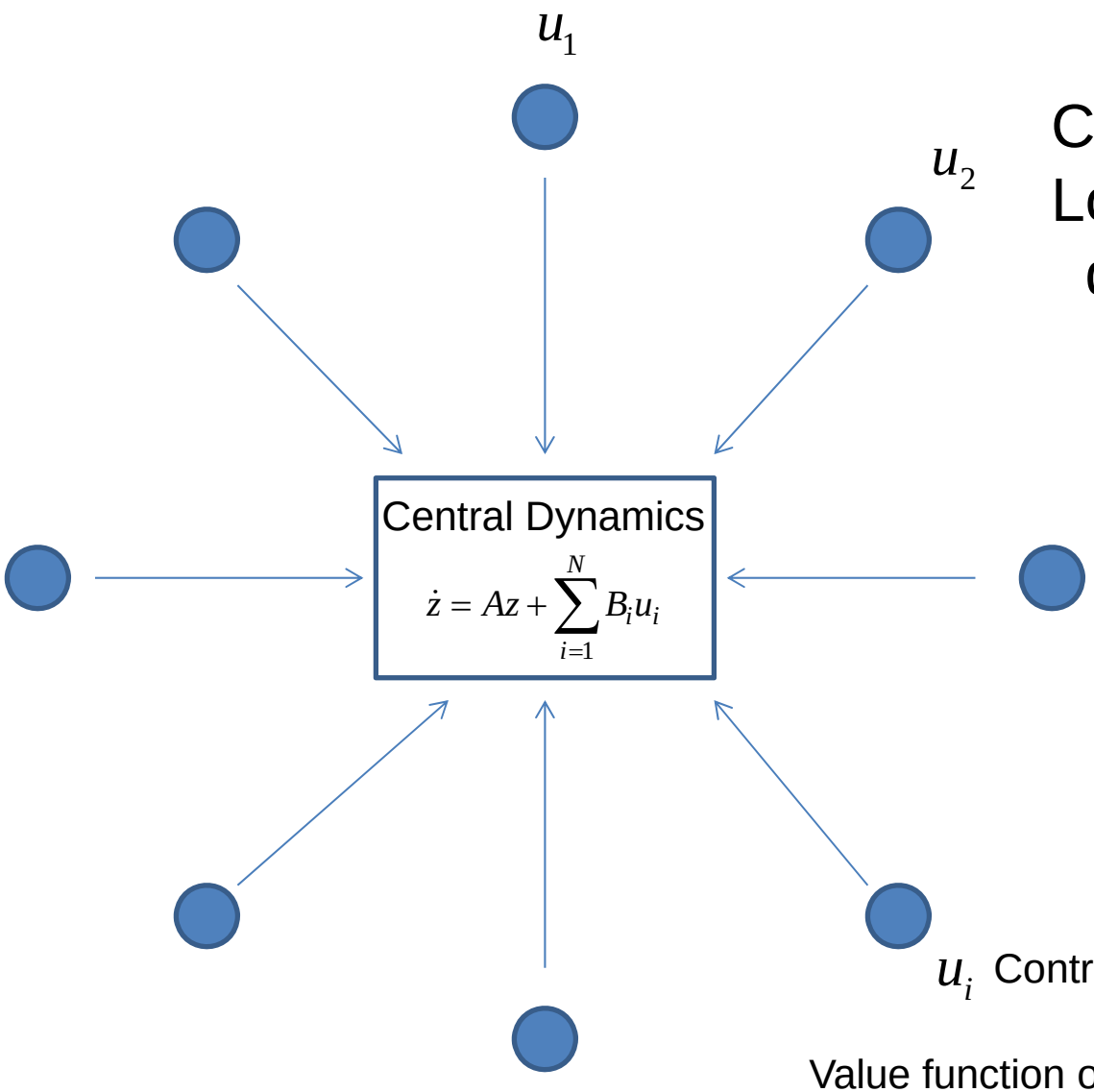
State dynamics of agent *i*

$$\dot{\delta}_i = A\delta_i + (d_i + g_i)B_i u_i - \sum_{j \in N_i} e_{ij} B_j u_j$$

Value function of player *i*

$$J_i(\delta_i(0), u_i, u_{-i}) = \tfrac{1}{2}\int_0^\infty (\delta_i^T Q_{ii}\delta_i + u_i^T R_{ii} u_i + \sum_{j \in N_i} u_j^T R_{ij} u_j)\, dt$$

Standard Multi-Agent Differential Game



Central Dynamics
Local Value Function
depends on ALL
other control actions

u_i Control action of player *i*

Value function of player *i*

$$J_i(z(0), u_i, u_{-i}) = \frac{1}{2}\int_0^{\infty} (z^T Q z + \sum_{j=1}^{N} u_j^T R_{ij} u_j)\, dt$$

Team Interest vs. Self Interest

Cooperation vs. Collaboration

The objective functions of each player can be written as a *team average* term plus a *conflict of interest* term:

$$J_1 = \tfrac{1}{3}(J_1 + J_2 + J_3) + \tfrac{1}{3}(J_1 - J_2) + \tfrac{1}{3}(J_1 - J_3) \equiv J_{team} + J_1^{coi}$$

$$J_2 = \tfrac{1}{3}(J_1 + J_2 + J_3) + \tfrac{1}{3}(J_2 - J_1) + \tfrac{1}{3}(J_2 - J_3) \equiv J_{team} + J_2^{coi}$$

$$J_3 = \tfrac{1}{3}(J_1 + J_2 + J_3) + \tfrac{1}{3}(J_3 - J_1) + \tfrac{1}{3}(J_3 - J_2) \equiv J_{team} + J_3^{coi}$$

For N-players

$$J_i = \tfrac{1}{N}\sum_{j=1}^{N} J_j + \tfrac{1}{N}\sum_{j=1}^{N}(J_i - J_j) \equiv J_{team} + J_i^{coi}, \quad i = 1, N$$

For *N-player zero-sum games*, the first term is zero, i.e. the players have no goals in common.

Problems with Nash Equilibrium Definition on Graphical Games

Game objective

$$V_i^*(\delta_i(t)) = \min_{u_i} \int_t^\infty \tfrac{1}{2}(\delta_i^T Q_{ii}\delta_i + u_i^T R_{ii} u_i + \sum_{j\in N_i} u_j^T R_{ij} u_j)\, dt$$

Define $u_{-i}(t), \quad \{u_j : j \in N_i\}$ Neighbors of node *i*

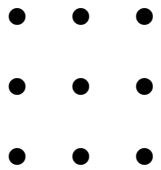
$u_{G-i} = \{u_j : j \in N, j \neq i\}$ All other nodes in graph

Def: Nash equilibrium

$\{u_1^*, u_2^*, \ldots, u_N^*\}$ are in Nash equilibrium if

$$J_i^* \triangleq J_i(u_i^*, u_{G-i}^*) \leq J_i(u_i, u_{G-i}^*), \ \forall i \in N$$

Counterexample. Disconnected graph



Then, each agent's cost does not depend on any other agent

$$J_i(u_i) = J_i(u_i, u_{G-i}) = J_i(u_i, u'_{G-i}), \ \forall i$$

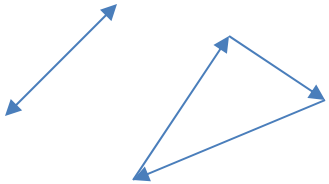
Let each node play his optimal control

$$J_i^* = J_i(u_i^*)$$

Then all agents are in Nash equilibrium

Note- this Nash is also coalition-proof

Another example



New Definition of Nash Equilibrium for Graphical Games

Def. Local Best response.

u_i^* is said to be agent *i's* local best response to fixed policies u_{-i} of its neighbors if

$$J_i(u_i^*, u_{-i}) \leq J_i(u_i, u_{-i}), \quad \forall u_i$$

Def: Interactive Nash equilibrium

$\{u_1^*, u_2^*, \ldots, u_N^*\}$ are in Interactive Nash equilibrium if

1. $J_i^* \triangleq J_i(u_i^*, u_{G-i}^*) \leq J_i(u_i, u_{G-i}^*), \ \forall i \in N$ i.e. they are in Nash equilibrium

2. There exists a policy u_j such that

$$J_i(u_j, u_{G-j}^*) \neq J_i(u_j^*, u_{G-j}^*), \quad \forall i, j \in N$$

That is, every player can find a policy that changes the value of every other player.

A restriction on what sorts of performance indices can be selected in multi-player graph games.

A condition on the reaction curves (Basar and Olsder) of the agents

This rules out the disconnected counterexample.

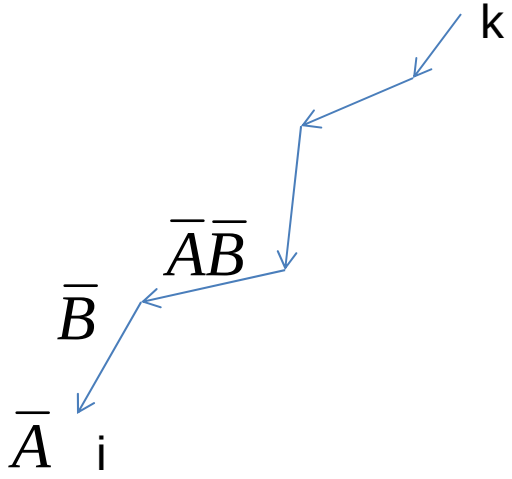
Theorem 3. Let (A,B_i) be reachable for all i.
Let agent i be in local best response

$$J_i\,(u_i^*, u_{-i}) \le J_i\,(u_i, u_{-i}), \quad \forall i$$

Then $\{u_1^*, u_2^*, \ldots, u_N^*\}$ are in global Interactive Nash iff the graph is strongly connected.

$$u_i = u_i(V_i) \equiv -(d_i + g_i) R_{ii}^{-1} B_i^T \frac{\partial V_i}{\partial \delta_i} \equiv -K_i\, p_i$$

$$u_k = -K_k\, p_k - v_k$$



Hamiltonian System

$$\begin{bmatrix} \dot{\delta} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} (I_N \otimes A) & ((L+G)\otimes I_n) diag(B_i K_i) \\ -diag(Q_{ii}) & -(I_N \otimes A^T) \end{bmatrix} \begin{bmatrix} \delta \\ p \end{bmatrix} + \begin{bmatrix} ((L+G)\otimes I_n)\underline{B}_k \\ 0 \end{bmatrix} \bar{v}_k \equiv \bar{A} \begin{bmatrix} \delta \\ p \end{bmatrix} + \bar{B}\bar{v}_k$$

$$\begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \bar{A}^2\bar{B} & \cdots \end{bmatrix}$$

Picks out the shortest path from node k to node i

Graphical Game Solution Equations

Value function

$$V_i(\delta_i(t)) = \tfrac{1}{2}\int_t^{\infty} (\delta_i^T Q_{ii}\delta_i + u_i^T R_{ii} u_i + \sum_{j\in N_i} u_j^T R_{ij} u_j)\, dt$$

Differential equivalent (Leibniz formula) is Bellman's Equation

$$H_i(\delta_i, \frac{\partial V_i}{\partial \delta_i}, u_i, u_{-i}) \equiv \frac{\partial V_i}{\partial \delta_i}^T \left(A\delta_i + (d_i + g_i) B_i u_i - \sum_{j\in N_i} e_{ij} B_j u_j \right) + \tfrac{1}{2}\delta_i^T Q_{ii}\delta_i + \tfrac{1}{2} u_i^T R_{ii} u_i + \tfrac{1}{2}\sum_{j\in N_i} u_j^T R_{ij} u_j = 0$$

Stationarity Condition

$$0 = \frac{\partial H_i}{\partial u_i} \quad \Rightarrow \quad u_i = -(d_i + g_i) R_{ii}^{-1} B_i^T \frac{\partial V_i}{\partial \delta_i}$$

1. Coupled HJ equations

$$\frac{\partial V_i}{\partial \delta_i}^T A_i^c + \tfrac{1}{2}\delta_i^T Q_{ii}\delta_i + \tfrac{1}{2}(d_i + g_i)^2 \frac{\partial V_i}{\partial \delta_i}^T B_i R_{ii}^{-1} B_i^T \frac{\partial V_i}{\partial \delta_i} + \tfrac{1}{2}\sum_{j\in N_i} (d_j + g_j)^2 \frac{\partial V_j}{\partial \delta_j}^T B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T \frac{\partial V_j}{\partial \delta_j} = 0, i \in N$$

$$H_i(\delta_i, \frac{\partial V_i}{\partial \delta_i}, u_i^*, u_{-i}^*) = 0$$

where $A_i^c = A\delta_i - (d_i + g_i)^2 B_i R_{ii}^{-1} B_i^T \frac{\partial V_i}{\partial \delta_i} + \sum_{j\in N_i} e_{ij}(d_j + g_j) B_j R_{jj}^{-1} B_j^T \frac{\partial V_j}{\partial \delta_j}, i \in N$

---

2. Best Response HJ Equations – other players have fixed policies u_j

$$0 = H_i(\delta_i, \frac{\partial V_i}{\partial \delta_i}, u_i^*, u_{-i}) \equiv \frac{\partial V_i}{\partial \delta_i}^T A_i^c + \tfrac{1}{2}\delta_i^T Q_{ii}\delta_i + \tfrac{1}{2}(d_i + g_i)^2 \frac{\partial V_i}{\partial \delta_i}^T B_i R_{ii}^{-1} B_i^T \frac{\partial V_i}{\partial \delta_i} + \tfrac{1}{2}\sum_{j\in N_i} u_j^T R_{ij} u_j$$

where $A_i^c = A\delta_i - (d_i + g_i)^2 B_i R_{ii}^{-1} B_i^T \frac{\partial V_i}{\partial \delta_i} - \sum_{j\in N_i} e_{ij} B_j u_j$

Theorem 1. Stability and Solution for Cooperative Nash Equilibrium.

Let $V_i > 0 \in C^1$, $i \in N$ be smooth solutions to HJ equations (23) and control policies u_i^*, $i \in N$ be given by (22) in terms of these solutions V_i. Then

a. Systems (8) are asymptotically stable.

b. u_i^*, u_{-i}^* are in cooperative Nash equilibrium and the corresponding game values are

$$J_i^*(\delta_i(0)) = V_i \ , i \in N \qquad (34)$$

Kyriakos Vamvoudakis

Theorem 2. Solution for Best Response Policy

Given fixed neighbor policies $u_{-i} = \{u_j : j \in N_i\}$, assume there is an admissible policy u_i. Let $V_i > 0 \in C^1$ be a smooth solution to the best response HJ equation (36) and let control policy u_i^* be given by (22) in terms of this solution V_i. Then

a. System (8) is asymptotically stable.

b. u_i^* is the best response to the fixed policies u_{-i} of its neighbors.

Online Solution of Graphical Games

Kyriakos Vamvoudakis

Use Reinforcement Learning

POLICY ITERATION

Algorithm 1. Policy Iteration (PI) Solution for N-player distributed games.

Step 0: Start with admissible initial policies $u_i^0, \forall i$.

Step 1: (Policy Evaluation) Solve for V_i^k using (14)

$$H_i(\delta_i, \frac{\partial V_i}{\partial \delta_i}^k, u_i^k, u_{-i}^k) = 0, \forall i = 1, \ldots, N \qquad (38)$$

Step 2: (Policy Improvement) Update the N-tuple of control policies using

$$u_i^{k+1} = \arg\min_{u_i} H_i(\delta_i, \frac{\partial V_i}{\partial \delta_i}^k, u_i, u_{-i}^k), \forall i = 1, \ldots, N$$

which explicitly is

$$u_i^{k+1} = -(d_i + g_i) R_{ii}^{-1} B_i^T \frac{\partial V_i}{\partial \delta_i}^k, \forall i = 1, \ldots, N. \qquad (39)$$

Go to step 1.

On convergence End ■

Convergence Results

Theorem 3. Convergence of Policy Iteration algorithm when only i^{th} agent updates its policy and all players u_{-i} in the neighborhood do not change. Given fixed neighbors policies u_{-i}, assume there exists an admissible policy u_i. Assume that agent i performs Algorithm 1 and the its neighbors do not update their control policies. Then the algorithm converges to the best response u_i to policies u_{-i} of the neighbors and to the solution V_i to the best response HJ equation (36).

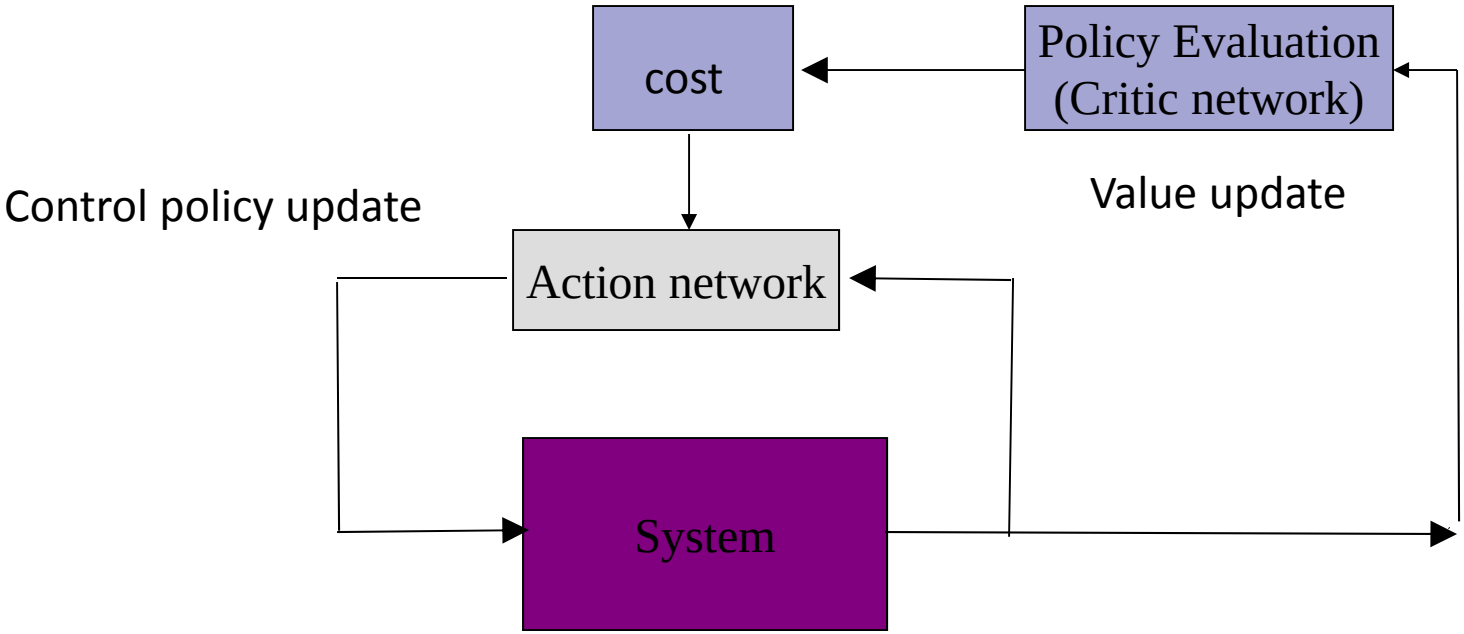
The next result concerns the case where all nodes update their policies at each step of the algorithm. Define the relative control weighting as $\rho_{ij} = \bar{\sigma}(R_{jj}^{-1} R_{ij})$, where $\bar{\sigma}(R_{jj}^{-1} R_{ij})$ is the maximum singular value of $R_{jj}^{-1} R_{ij}$.

Theorem 4. Convergence of Policy Iteration algorithm when all agents update their policies. Assume all nodes i update their policies at each iteration of PI. Then for small enough edge weights e_{ij} and ρ_{ij}, μ_i converges to the global Nash equilibrium and for all i, and the values converge to the optimal game values $V_i^k \to V_i^*$.

Online Solution of Graphical Games

New Structure of Adaptive Controller

Reinforcement Learning Adaptive Critic



Critic and Actor tuned simultaneously

Leads to ONLINE FORWARD-IN-TIME implementation of optimal control

Do not need to know system drift dynamics $\dot{x}_i = f(x_i) + g(x_i)u_i$

Best Paper Award, Int. Joint Conf. Neural Networks, Barcelona, 2010. **Draguna Vrabie** and F.L. Lewis, "Adaptive Dynamic Programming Algorithm for Finding Online the Equilibrium Solution of the Two-Player Zero-Sum Differential Game."

F.L. Lewis and D. Vrabie, "Reinforcement learning and adaptive dynamic programming for feedback control," IEEE Circuits & Systems Magazine, Invited Feature Article, pp. 32-50, Third Quarter 2009.

6. ADP Using Reduced State Information (Output Feedback) (Partially Observable Markov Decision Processes)

DT Linear Quadratic Regulator Optimal Control

DT System $x_{k+1} = Ax_k + Bu_k$

$y_k = Cx_k$

Performance measure $V^{\mu}(x_k) = \sum_{i=k}^{\infty} \left(y_i^T Q y_i + u_i^T R u_i \right) \equiv \sum_{i=k}^{\infty} r_i$

Utility $r_k = y_k^T Q y_k + u_k^T R u_k$

Value is quadratic in the state $V^{\mu}(x_k) = x_k^T P x_k$

Algebraic Riccati Equation

$$0 = A^T PA - P + C^T QC - A^T PB(R + B^T PB)^{-1} B^T PA$$

Optimal feedback gain (policy)

$$u_k = -Kx_k = -(R + B^T PB)^{-1} B^T PAx_k$$

Off-line solution

Requires knowledge of system dynamics *A,B,C*

We want online solution of ARE using only measured input/output data

Bellman equation $x_k^T P x_k = y_k^T Q y_k + u_k^T R u_k + x_{k+1}^T P x_{k+1}$

Value function is Quadratic in the state

Policy Iteration Algorithm

i. Start with stabilizing control policy

ii. Value Update $0 = -x_k^T P^{j+1} x_k + y_k^T Q y_k + (u_k^j)^T R u_k^j + x_{k+1}^T P^{j+1} x_{k+1}$ Lyapunov Equation

iii. Policy Improvement $u_k^{j+1} = -(R + B^T P^{j+1} B)^{-1} B^T P^{j+1} A x_k$

Value Iteration Algorithm

i. Start with ANY control policy

ii. Value Update $x_k^T P^{j+1} x_k = y_k^T Q y_k + (u_k^j)^T R u_k^j + x_{k+1}^T P^j x_{k+1}$ Lyapunov recursion

iii. Policy Improvement $u_k^{j+1} = -(R + B^T P^{j+1} B)^{-1} B^T P^{j+1} A x_k$

Requires state measurements

Expanded State Equation (ESE)

Express state in terms of inputs and outputs

System $x_{k+1} = Ax_k + Bu_k \qquad x_k \in R^n,\ u_k \in R^m,\ y_k \in R^p$

$y_k = Cx_k$

U_N Reachability matrix

Expanded State Equation

$$x_k = A^N x_{k-N} + \begin{bmatrix} B & AB & A^2B & \cdots & A^{N-1}B \end{bmatrix} \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-N} \end{bmatrix} \quad \bar{u}_{k-1,k-N}$$

$$\bar{y}_{k-1,k-N} \begin{bmatrix} y_{k-1} \\ y_{k-2} \\ \vdots \\ y_{k-N} \end{bmatrix} = \begin{bmatrix} CA^{N-1} \\ \vdots \\ CA \\ C \end{bmatrix} x_{k-N} + \begin{bmatrix} 0 & CB & CAB & \cdots & CA^{N-2}B \\ 0 & 0 & CB & \cdots & CA^{N-3}B \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & & 0 & CB \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-N} \end{bmatrix}$$

V_N Observability matrix

T_N Invertibility matrix of Markov Parameters

$$x_k = A^N x_{k-N} + U_N \bar{u}_{k-1,k-N}$$

$$\bar{y}_{k-1,k-N} = V_N x_{k-N} + T_N \bar{u}_{k-1,k-N}$$

Express State in Terms of Previous Inputs & Outputs

Observable implies V_N has full column rank n

$A^N = MV_N$ for N greater than the observability index

$$M = A^N V_N^+ + Z(I - V_N V_N^+) \equiv M_0 + M_1$$

MP pseudoinverse is $V_N^+ = (V_N^T V_N)^{-1} V_N^T$

Projection on range perp. V_N *is* $P(R^\perp(V_N)) = I - V_N V_N^+$

1. From $\bar{y}_{k-1,k-N} = V_N x_{k-N} + T_N \bar{u}_{k-1,k-N}$

$$A^N x_{k-N} = MV_N x_{k-N} = M\bar{y}_{k-1,k-N} - MT_N \bar{u}_{k-1,k-N}$$

$$(M_0 + M_1)V_N x_{k-N} = (M_0 + M_1)\bar{y}_{k-1,k-N} - (M_0 + M_1)T_N \bar{u}_{k-1,k-N}$$

but $M_1 V_N = 0$

so $0 = M_1 \bar{y}_{k-1,k-N} - M_1 T_N \bar{u}_{k-1,k-N}, \quad \forall M_1 \; s.t. \; M_1 V_N = 0$

Then $A^N x_{k-N} = M_0 V_N x_{k-N} = M_0 \bar{y}_{k-1,k-N} - M_0 T_N \bar{u}_{k-1,k-N}$

2. From $x_k = A^N x_{k-N} + U_N \bar{u}_{k-1,k-N}$

Then state in terms of inputs & outputs is

$$x_k = M_0 \bar{y}_{k-1,k-N} + (U_N - M_0 T_N)\bar{u}_{k-1,k-N} \equiv M_y \bar{y}_{k-1,k-N} + M_u \bar{u}_{k-1,k-N}$$

Markov parameters

So

$$x_k = \begin{bmatrix} M_u & M_y \end{bmatrix} \begin{bmatrix} \bar{u}_{k-1,k-N} \\ \bar{y}_{k-1,k-N} \end{bmatrix}$$

$$\bar{y}_{k-1,k-N} = \begin{bmatrix} y_{k-1} \\ y_{k-2} \\ \vdots \\ y_{k-N} \end{bmatrix}, \qquad \bar{u}_{k-1,k-N} = \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-N} \end{bmatrix}$$

Prior Work

$$x_k = M_0 \bar{y}_{k-1,k-N} + \left(U_N - M_0 T_N\right)\bar{u}_{k-1,k-N} \quad \equiv M_y \bar{y}_{k-1,k-N} + M_u \bar{u}_{k-1,k-N}$$

But this needs to know dynamics *A,B,C* to compute M_y and M_u

A lot of work has been done to

express the optimal control policy $u_k = -Kx_k = -(R + B^T PB)^{-1} B^T PAx_k$

in terms of inputs and outputs

and identify the Markov Parameters online

We can avoid all this by using Reinforcement Learning techniques

RL Can learn the System Parameters online

Express Bellman Equation in Terms of Inputs & Outputs

Bellman Equation $x_k^T P x_k = y_k^T Q y_k + u_k^T R u_k + x_{k+1}^T P x_{k+1}$ Quadratic in state

We know $x_k = \begin{bmatrix} M_u & M_y \end{bmatrix} \begin{bmatrix} \bar{u}_{k-1,k-N} \\ \bar{y}_{k-1,k-N} \end{bmatrix} \equiv \bar{z}_{k-1,k-N}$

$$\bar{y}_{k-1,k-N} = \begin{bmatrix} y_{k-1} \\ y_{k-2} \\ \vdots \\ y_{k-N} \end{bmatrix}, \quad \bar{u}_{k-1,k-N} = \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-N} \end{bmatrix}$$

Value in terms of i/o $V^u(x_k) = x_k^T P x_k = \bar{z}_{k-1,k-N}^T \begin{bmatrix} M_u^T \\ M_y^T \end{bmatrix} P \begin{bmatrix} M_u & M_y \end{bmatrix} \bar{z}_{k-1,k-N}$

$$V^u(x_k) = \bar{z}_{k-1,k-N}^T \begin{bmatrix} M_u^T P M_u & M_u^T P M_y \\ M_y^T P M_u & M_y^T P M_y \end{bmatrix} \bar{z}_{k-1,k-N} \equiv \bar{z}_{k-1,k-N}^T \bar{P}\, \bar{z}_{k-1,k-N}$$

ARE solution and Markov Parameters

Bellman Equation $\bar{z}_{k-1,k-N}^T \bar{P} \bar{z}_{k-1,k-N} = y_k^T Q y_k + u_k^T R u_k + \bar{z}_{k,k-N+1}^T \bar{P} \bar{z}_{k,k-N+1}$

TD error $e_k = -\bar{z}_{k-1,k-N}^T \bar{P} \bar{z}_{k-1,k-N} + y_k^T Q y_k + u_k^T R u_k + \bar{z}_{k,k-N+1}^T \bar{P} \bar{z}_{k,k-N+1}$

Quadratic in previous inputs & outputs

We can use either PI or VI to learn the parameter matrix $\bar{P}$
ONLINE in Real-Time using measurements of inputs/outputs
Along the system trajectories

System parameters are not needed for Value Update Step

Policy Update Step with no System Information

Bellman Equation $$\bar{z}_{k-1,k-N}^T \bar{P} \bar{z}_{k-1,k-N} = y_k^T Q y_k + u_k^T R u_k + \bar{z}_{k,k-N+1}^T \bar{P} \bar{z}_{k,k-N+1}$$

Policy Update $$u(x_k) = \arg\min_{u_k} \left(y_k^T Q y_k + u_k^T R u_k + \bar{z}_{k,k-N+1}^T \bar{P} \bar{z}_{k,k-N+1} \right)$$

Value is quadratic in previous i/o $$V^u(x_k) = \bar{z}_{k-1,k-N}^T \begin{bmatrix} M_u^T P M_u & M_u^T P M_y \\ M_y^T P M_u & M_y^T P M_y \end{bmatrix} \bar{z}_{k-1,k-N} \equiv \bar{z}_{k-1,k-N}^T \bar{P} \, \bar{z}_{k-1,k-N}$$

$$\bar{z}_{k-1,k-N} = \begin{bmatrix} \bar{u}_{k-1,k-N} \\ \bar{y}_{k-1,k-N} \end{bmatrix}$$

Partition as $$\bar{z}_{k,k-N+1}^T \bar{P} \bar{z}_{k,k-N+1} = \begin{bmatrix} u_k \\ \bar{u}_{k-1,k-N+1} \\ \bar{y}_{k,k-N+1} \end{bmatrix}^T \begin{bmatrix} p_0 & p_u & p_y \\ p_u^T & P_{22} & P_{23} \\ p_y^T & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} u_k \\ \bar{u}_{k-1,k-N+1} \\ \bar{y}_{k,k-N+1} \end{bmatrix}$$

Quadratic in inputs and outputs.
Has same form as Q learning.

Policy Update Step $$u(x_k) = \arg\min_{u_k} \left(y_k^T Q y_k + u_k^T R u_k + \begin{bmatrix} u_k \\ \bar{u}_{k-1,k-N+1} \\ \bar{y}_{k,k-N+1} \end{bmatrix}^T \begin{bmatrix} p_0 & p_u & p_y \\ p_u^T & P_{22} & P_{23} \\ p_y^T & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} u_k \\ \bar{u}_{k-1,k-N+1} \\ \bar{y}_{k,k-N+1} \end{bmatrix} \right)$$

Differentiate wrt u_k $$0 = R u_k + p_0 u_k + p_u \bar{u}_{k-1,k-N+1} + p_y \bar{y}_{k,k-N+1}$$

Policy Update $$u_k = -(R + p_0)^{-1} \left(p_u \bar{u}_{k-1,k-N+1} + p_y \bar{y}_{k,k-N+1} \right)$$

Do NOT NEED *A* or *B!*

Compare to $$u_k^{j+1} = -(R + B^T P^{j+1} B)^{-1} B^T P^{j+1} A x_k$$

Policy Iteration using output measurements

Policy Evaluation- solve Bellman equation for $\bar{P}$

$$\bar{z}_{k-1,k-N}^T \bar{P} \bar{z}_{k-1,k-N} = y_k^T Q y_k + u_k^T R u_k + \bar{z}_{k,k-N+1}^T \bar{P} \bar{z}_{k,k-N+1}$$

Unpack parameters into matrix form

$$\bar{z}_{k,k-N+1}^T \bar{P} \bar{z}_{k,k-N+1} = \begin{bmatrix} u_k \\ \bar{u}_{k-1,k-N+1} \\ \bar{y}_{k,k-N+1} \end{bmatrix}^T \begin{bmatrix} p_0 & p_u & p_y \\ p_u^T & P_{22} & P_{23} \\ p_y^T & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} u_k \\ \bar{u}_{k-1,k-N+1} \\ \bar{y}_{k,k-N+1} \end{bmatrix}$$

Control Update

$$u_k = -(R + p_0)^{-1} \left(p_u \bar{u}_{k-1,k-N+1} + p_y \bar{y}_{k,k-N+1} \right)$$

Does not need ANY system dynamics
Looks a lot like Q learning – but Q needs states

$$\bar{y}_{k-1,k-N} = \begin{bmatrix} y_{k-1} \\ y_{k-2} \\ \vdots \\ y_{k-N} \end{bmatrix}, \quad \bar{u}_{k-1,k-N} = \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-N} \end{bmatrix}$$

The Controller is in ARMA Polynomial Regulator Form!

$$u_k = -(R + p_0)^{-1}\left(p_u \bar{u}_{k-1,k-N+1} + p_y \bar{y}_{k,k-N+1} \right)$$

An ARMA Controller that is equivalent to the optimal SVFB gain

Compare to the Optimal Polynomial regulator in
Lewis and Syrmos, Optimal Control, 1995

Simulation Example

Also works for unstable systems

$$x_{k+1} = \begin{bmatrix} 1.1 & -0.3 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & -0.8 \end{bmatrix} x_k$$

ARE solution

$$P = \begin{bmatrix} 1.0150 & -0.8150 \\ -0.8150 & 0.6552 \end{bmatrix}$$

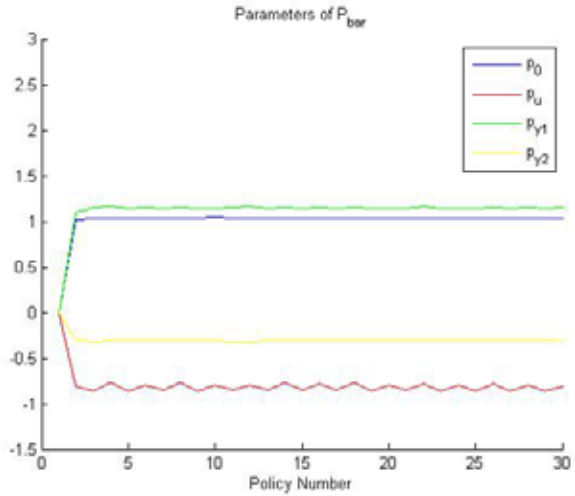
Actual Pbar matrix

$$\bar{P} \equiv \begin{bmatrix} M_u^T P M_u & M_u^T P M_y \\ M_y^T P M_u & M_y^T P M_y \end{bmatrix}$$

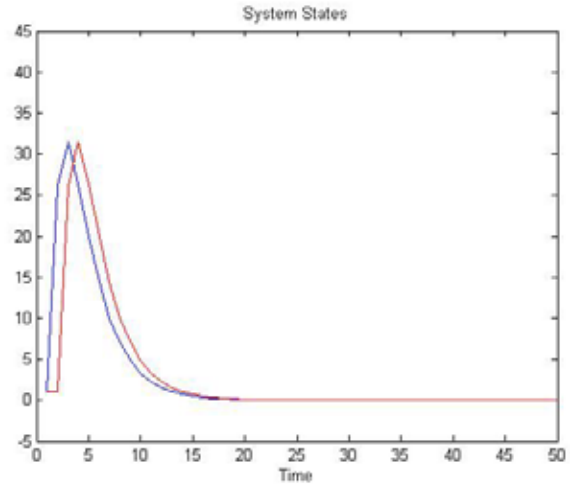
$$\bar{P} = \begin{bmatrix} 1.0150 & -0.8440 & 1.1455 & -0.3165 \\ -0.8440 & 0.7918 & -1.0341 & 0.2969 \\ 1.1455 & -1.0341 & 1.3667 & -0.3878 \\ -0.3165 & 0.2969 & -0.3878 & 0.1113 \end{bmatrix}$$

Learned Pbar matrix

$$\hat{\bar{P}} = \begin{bmatrix} 1.1340 & -0.8643 & 1.1571 & -0.3161 \\ -0.8643 & 0.7942 & -1.0348 & 0.2966 \\ 1.1571 & -1.0348 & 1.3609 & -0.3850 \\ -0.3161 & 0.2966 & -0.3850 & 0.1102 \end{bmatrix}$$



Convergence of p_0, p_u, p_y



State trajectories

$$0 = A^T PA - P + C^T QC - A^T PB(R + B^T PB)^{-1} B^T PA$$

Solves ARE online

Without knowing *A*, *B* and without measuring states

Our revels now are ended. These our actors,
As I foretold you, were all spirits, and
Are melted into air, into thin air.

The cloud-capped towers, the gorgeous palaces,
The solemn temples, the great globe itself,
Yea, all which it inherit, shall dissolve,
And, like this insubstantial pageant faded,
Leave not a rack behind.

We are such stuff as dreams are made on,
and our little life is rounded with a sleep.

Prospero, in The Tempest, act 4, sc. 1, l. 152-6, Shakespeare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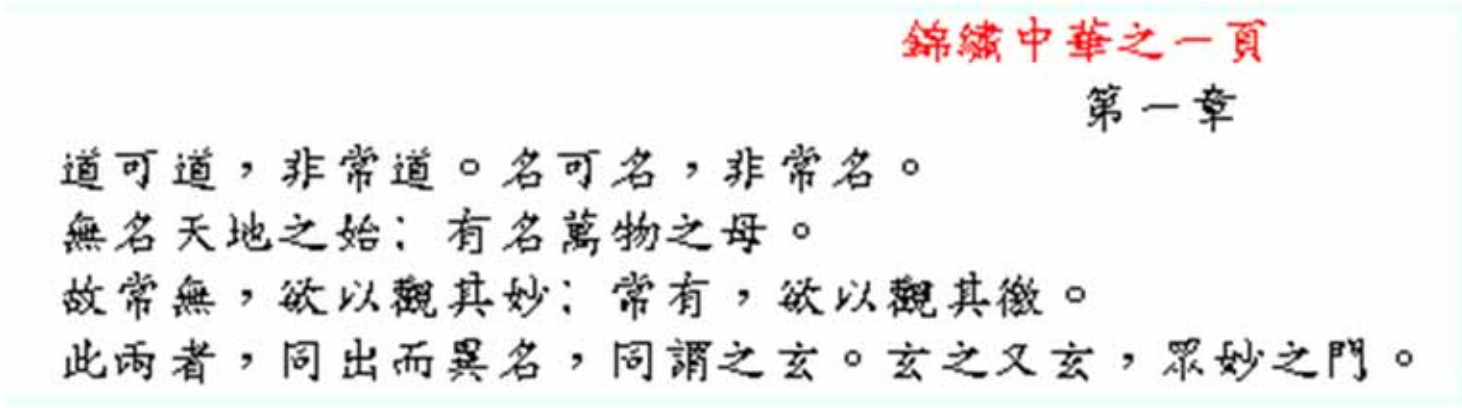
錦繡中華之一頁

第一章

道可道，非常道。名可名，非常名。
無名天地之始；有名萬物之母。
故常無，欲以觀其妙；常有，欲以觀其徼。
此兩者，同出而異名，同謂之玄。玄之又玄，眾妙之門。

The way that can be told is not the Constant Way
The name that can be named is not the Constant Name

For nameless is the true way
Beyond the myriad experiences of the world

To experience without intention is to sense the world

All experience is an arch
wherethrough gleams that untravelled land
whose margins fade forever as we move

Dao ke dao feichang dao
Ming ke ming feichang ming